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Application of Transfer Matrix Method in Acoustics

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Abstract: This paper shows the application of the Transfer Matrix Method in the Acoustics Module of COMSOL Multiphysics. The analyzed system is the combustion chamber of an industrial gas turbine and in the 3D acoustic simulation the burners are geometrically cut and substituted by transfer matrices. The application of this methodology allows us to take into account parameters which can not be considered in the Acoustics module (e.g. mean flow and viscosity) and to reduce the computational effort.

The Transfer Matrix method is used when the whole system can be represented into a sequence of subsystems that interact only with adjacent subsystems. This method is frequently used in mathematics and, particularly, in acoustics. The application of the present method is really useful, because it permits to break the total system into a network of subsystems, linked together by transfer matrices. These matrices can be obtained analytically, experimentally or numerically. The great importance of this method is the possibility of using data from experiments or from detailed numerical simulations inside the transfer matrix.

Keywords: transfer matrix, combustion instability, gas turbine, acoustics, wave propagation, eigenvalues.

1 Introduction

The application of the Transfer Matrix method is common in mathematics and in physics, particularly in acoustics. This method is really useful when it is possible to break the examined system into a network of subsystems, linked together by transfer matrices. The construction of these matrices can be obtained analitically, experimentally or numerically, and literature is full of these types of applications in a wide set of fields. The large use of this method lies in the possibility of using data coming from experiments or from detailed numerical simulations inside the transfer matrix.

One application field is thermoacoustics in gas turbine combustion chamber. This method is widely used in the Acoustics Network, a method which uses mathematical transfer matrices to connect each other lumped acoustic elements, so that there are a lot of different approaches adopted in order to obtain a valid correlation between the acoustic properties of each element (e.g. [9], [6], [1], [7]).

An hybrid technique has been proposed ([4]-[2]), combining both lumped elements and three-dimensional elements. In this way the advantages of the single techniques are exploited, such as the simplicity of the lumped acoustic elements. A numerical procedure able to detect the complex eigenvalues of the analyzed system in presence of a flame transfer function and a matrix transfer function is implemented within COMSOL Multiphysics.

In this paper this coupling is shown looking at the simple application of the transfer matrix inside COMSOL Multiphysics. Some test cases of simple geometries are first examined, with and without a burner transfer matrix (BTM), so that it is possible to compare the results. A simple cylindrical combustor with different configurations has been analyzed. In a subsequent step the potentiality of this method has been examined, looking at the influence of velocity and viscosity inside the burner.

Summarizing, in the first part of this work we want to compare the results with and without burner transfer matrix in order to show that this method follows the physics of the phenomenon. In the second part the potentiality of this approach are presented.

2 MATHEMATICAL MODEL

In gas turbine combustion chamber Mach number is generally far below unity, so the flow velocity can be regarded as negligible, except in some areas, such as conduits of the burner. In terms of propagation of pressure waves, these areas can be treated as compact elements modelled by means of specific transfer function matrices, obtained experimentally or numerically through CFD or aeroacoustics codes. As a consequence, the flow velocity can be neglected within the computational domain in comparison with the sound velocity. Moreover, the effects of viscous losses and heat transfer can be neglected. The fluid is treated as an ideal gas, so that the ratio of the specific heats is considered constant. Under such hypotheses, in presence of heat fluctuations, the inhomogeneous wave equation can be obtained

$$\frac{1}{\bar{c}^2}\frac{\partial^2 p'}{\partial t^2} - \bar{\rho}\nabla\cdot\left(\frac{1}{\bar{\rho}}\nabla p'\right) = \frac{\gamma-1}{\bar{c}^2}\frac{\partial q'}{\partial t} \quad (1)$$

where q' is fluctuation of the heat input per unit volume, the prime a perturbation and overbar denotes a time average mean value. The term at the RHS of Eq. (1) shows that the rate of non-stationary heat release creates a monopole source of acoustic pressure disturbance. Considering that mean flow velocity is neglected, no entropy wave is generated and the pressure fluctuations are related to the velocity fluctuations by

$$\frac{\partial \mathbf{u'}}{\partial t} + \frac{1}{\bar{\rho}}\nabla p' = 0. \quad (2)$$

Pressure, velocity and heat release fluctuations are expressed by complex functions of time and position

$$A' = \Re(\widehat{A}(\mathbf{x})\exp(i\omega t)) \quad (3)$$

where ω is complex frequency and A is the generic fluctuation. ω real part gives the frequency of oscillations f, while the imaginary part gives the growth rate g at which the amplitude of oscillations increases per cycle

$$f = \frac{\Re(\omega)}{2\pi} \qquad g = -\Im(\omega). \quad (4)$$

Then, using Eq. (1) and Eq. (3), the acoustic pressure waves are governed by the following equation

$$\frac{\lambda^2}{\bar{c}^2}\widehat{p} - \bar{\rho}\nabla\cdot\left(\frac{1}{\bar{\rho}}\nabla\widehat{p}\right) = -\frac{\gamma-1}{\bar{c}^2}\lambda\widehat{q} \quad (5)$$

where $\lambda = -i\omega$ and c is the velocity of the sound. Eq. (5) shows a quadratic eigenvalue problem which is solved by means of an iterative linearization procedure. COMSOL Multiphysics uses the ARPACK FORTRAN as numerical routines for large-scale eigenvalue problems. This code is based on a variant of the Arnoldi algorithm called the implicit restarted Arnoldi method. The solver reformulates the quadratic eigenvalue problem as a linear eigenvalue problem of the conventional form, and iteratively updates the linearization point until convergence is reached. Through this method we can model the heat release fluctuation as a variable of space as shown in Eq. (3).

2.1 Transfer Matrix Method

The mathematical model of the burners, expressed as lumped elements, is represented by a system of linear equations, which is the transfer matrix. In this system the unknown are the fluctuations of acoustic pressure p' and acoustic velocity u' at the junctions, or ports of the element. Instead of these acoustic variables, the Riemann invariants $\mathcal{F}$ and $\mathcal{G}$ are usually adopted. Riemann invariants are related to acoustic pressure and velocity as follows

$$\mathcal{F} = \frac{1}{2}\left(\frac{p'}{\rho c} + u'\right) \qquad \mathcal{G} = \frac{1}{2}\left(\frac{p'}{\rho c} - u'\right) \quad (6)$$

and they can be thought as waves propagating, respectively, in the downstream and upstream direction. For a simple duct of uniform cross section, under the hypotheses of plane waves without mean flow and dissipative effects, the transfer matrix is given by [8]

$$\begin{pmatrix}\mathcal{F}_d\\ \mathcal{G}_d\end{pmatrix} = \begin{pmatrix}\exp(ikl) & 0\\ 0 & \exp(-ikl)\end{pmatrix}\begin{pmatrix}\mathcal{F}_u\\ \mathcal{G}_u\end{pmatrix} \quad (7)$$

where $k = \omega/c$ is the wave number, $c = \sqrt{\gamma RT}$ is the speed of sound and l is the length of the duct. This kind of approach loses its validity when the burner has a complex geometry or if it is necessary to consider more effects. Therefore, the burner can be modelled as a compact element using a transfer matrix obtained through experimental data. Theory and derivations of these relations are well described in literature [8, 9, 10]. Assuming a one-dimensional flow and linearizing the mass and the conservation equations, it is obtained

$$\left[A\left(\frac{p'}{\rho c}M+u'\right)\right]_u^d=0 \qquad (8)$$

$$\frac{i\omega}{c}u'_u l_{eff}+\left[Mu'+\frac{p'}{\rho c}\right]_u^d+\zeta M_d u'_d=0 \qquad (9)$$

In Eq. (9) the effective length l_{eff} is a measure of the accelerated mass in the compact element

$$l_{eff}=\int_{x_u}^{x_d}\frac{A_u}{A(x)}\,dx \qquad (10)$$

and it takes into account the variation of section between plenum and burner. ζ is the acoustical losses and generally closed to the time mean flow loss coefficient. Using effective length l_{eff} and pressure loss coefficient ζ, neglecting higher order Mach number terms, the transfer matrix of a compact element is obtained from Eq. (8) and Eq. (9)

$$\begin{bmatrix}\frac{p'}{\rho c}\\ u'\end{bmatrix}_d=\begin{bmatrix}1 & M_u-\alpha M_d(1+\zeta)-ikl_{eff}\\ \alpha M_u-M_d & \alpha+M_d ikl_{eff}\end{bmatrix}\begin{bmatrix}\frac{p'}{\rho c}\\ u'\end{bmatrix}_u \qquad (11)$$

where $k=\omega/c$ is wave number and $\alpha=A_u/A_d$ is the area ratio.

3 Use of COMSOL Multiphysics

The problem is solved in the frequency domain using the eigenfrequency *Pressure Acoustics* application mode inside the *Acoustics Module*.

The application of the Transfer Matrix Method in COMSOL Multiphysics requires only few instructions. Once chosen what kind of transfer matrix to use, such as that in Eq. 7 or that in Eq. 11, it is necessary to identify the surfaces on the analyzed geometry which represent the matrix interfaces (as can be seen in Fig. 2). Transfer Matrix can be written as an equation system in `Global Equations`. As in this application the matrix correlates acoustic pressure and velocity, these are the reference quantities. In fact, the boundary conditions on the matrix interfaces are written as `Normal Acceleration` considering the acoustic velocity. Acoustic pressure on the matrix interfaces is written in `Integration Coupling Variables/Boundary Variables`.

4 TESTS ON LINEAR COMBUSTION CHAMBER

4.1 Simplified one-dimensional combustor

The analyzed scheme is characterized by three cylindrical ducts, that represent plenum, burner and combustion chamber, respectively. In this case, the burner has a sectional area lower than those of plenum and combustion chamber (Fig. 1), with a ratio between the length of the burner (y) and the overall length (L) equal to 0.1. It is assumed a choking at the plenum inlet and at the combustor exit, so that they act approximately like a closed end ($u'=0$).

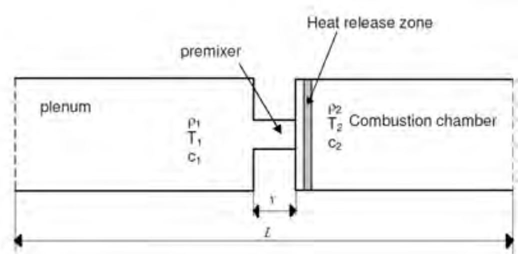

Figure 1: Scheme of a straight duct with variation of section.

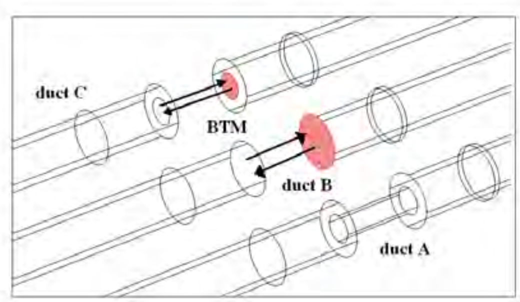

Figure 2: Computational grids of the straight ducts with variation of section (particular). Duct A is without BTM, duct B and duct C are with BTM.

Fig. 2 shows the criteria adopted in order to substitute the burner with a transfer matrix: duct A represents the original geometry, while ducts B and C represent the geometry with a transfer matrix after removing the volume of the burner. The difference between the schemes B and C concerns the surfaces adopted as interface between the transfer matrix (1D model) and the FEM analysis

(3D model). Such surfaces are highlighted by a coloured surface. In duct C the interfaces are circles with the same area of the burner, whereas in duct B the interfaces are the whole exit section from plenum and the whole inlet section to combustion chamber.

Heat release fluctuations are modelled considering that the rate of heat release fluctuations $q'(x,t)$ is supposed to be related to the local velocity u' upstream the flame zone with a time delay τ, following the approach used in [5, 4]

$$q'(x,t) = Q'(t)\delta(x-b) \tag{12}$$

$$Q'(t) = -\frac{\beta\overline{\rho c}^2}{\gamma-1}u'_1(t-\tau) \tag{13}$$

where $Q(t)$ is the rate of heat input per unit area of the cross section of the duct and subscript 1 denotes conditions just upstream of this region of heat input, that is $u'_1(t) = u'(x_1^-,t)$ (see Fig. 1). The Eq. (12) relates the fluctuations of heat input rate per unit volume, $q'(x,t)$, to the fluctuations of heat input rate per unit area of the cross section, $Q'(t)$, through the Dirac's delta $\delta(x-b)$. The non dimensional number β gives a measure of coupling between heat release and velocity fluctuations, while τ can be valued as the convection time from fuel injection to its combustion. In the FEM eigenvalue analysis, heat release fluctuations are supposed to occur in a thin volume with thickness s and the Dirac's delta $\delta(x-b)$ can be approximated by $1/s$.

Considering Eq. (3), setting $\lambda = -i\omega$ and simplifying, Eq. (5) becomes

$$\frac{1}{\overline{\rho c}^2}\lambda^2\widehat{p}-\frac{1}{\overline{\rho}}\nabla\left(\nabla\widehat{p}\right) = \left[-\beta\delta(x-b)(-\lambda)\widehat{u}(b^-)\exp(\lambda\tau)\right] \tag{14}$$

that is the governing equation to be solved in the internal domain by the FEM method. It is worth noting that the coefficient of intensity β and the time delay τ can be expressed as function of the frequency ω of the heat release fluctuations. For practical combustors the coefficients β and τ can be obtained from experiments or CFD calculations. In these tests, for arbitrary admissible values, β and τ are assumed.

Figg. 3- 4 shows that the application of analytical transfer matrix to a system with a variation of section gives very good results both in modeshape, such as in the identification of the correct complex eigenfrequencies (see Table 1).

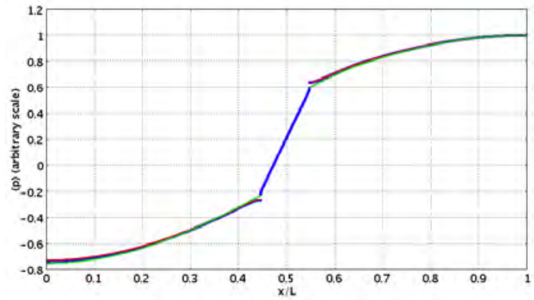

Figure 3: Modeshape corresponding to the first eigenmode of the straight duct with variation of section (Fig. 2) and $T_2/T_1 = 2$. Red line for duct C, green line for duct B, blue line for duct A.

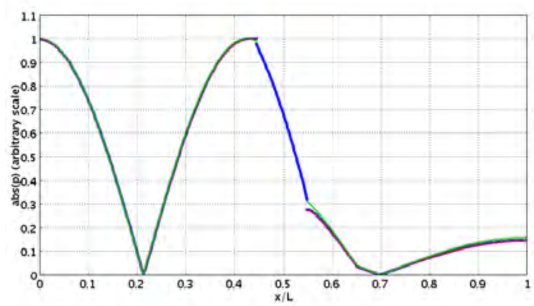

Figure 4: Modeshape corresponding to the second eigenmode of the straight duct with variation of section .

	1^{st} mode	2^{nd} mode
duct A	77.4 + 4.2i Hz	212.8 - 0.3i Hz
duct B	79.7 + 3.7i Hz	213.4 - 0.4i Hz
duct C	77.2 + 4.2i Hz	212.7 - 0.3i Hz

Table 1: Values of the eigenfrequencies.

Duct C gives results much better than duct B, so that three-dimensional effects are reproduced much better. These differences can be seen both in the modeshapes and in the value of the eigenfrequencies.

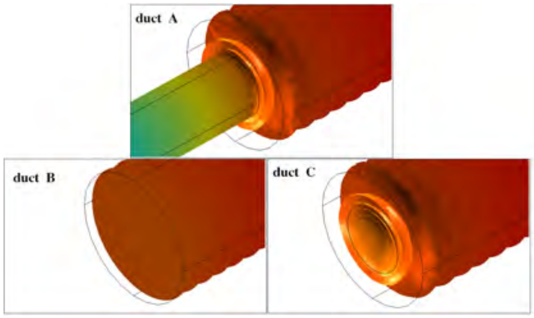

Figure 5: Acoustic pressure field for duct A, B and C.

Fig. 5 shows clearly the three-dimensional effects and the difference between the two cases. It appears that duct C succeeds in reproducing the same acoustic pressure field around the burner of the original one (duct A), whereas duct B does not recognize the correct variation of section due to the presence of the burner.

From the analysis of this test case, it appears that if the interfaces are correctly identified, the model with transfer matrix is able to reproduce very accurately the original system.

4.1.1 Experimental heat release and transfer matrix

In the previous tests a theoretical heat release law and an analytical transfer matrix have been considered. In the following tests heat release has been described through experimental data taken from [6, 1]. This expression is based on the assumption that heat release fluctuations are linked to the velocity fluctuations upstream of the flame through the following equation:

$$F_Q(\omega) = \frac{\dot{Q}'/\overline{\dot{Q}}}{u'_c/\overline{u}_c} \tag{15}$$

Following [11] the flame transfer function can be modelled as

$$F_Q(\omega) = n\left(\frac{T_h}{T_c} - 1\right)^{-1} \exp(-i\omega\tau - 0.5\omega^2\sigma^2) \tag{16}$$

where n is the interaction index defining the intensity of the coupling between heat release and velocity fluctuations. σ is the time lag distribution, which considers the variation of flame length.

The flame transfer function (Eq. (15)) and the transfer matrix of a compact element (Eq. (11)) have been applied to the simplified linear combustion chamber with variation of section (Fig. 1). Even in this case, transfer matrix has been applied removing the burner, so that the upstream port of the matrix is the exit from the plenum and the downstream is the inlet of the combustion chamber. Flame transfer function has been placed at the beginning of the combustion chamber, like shown in Fig. 1. The operating conditions have been set following the experimental values given in [6], above all for the values of τ and σ for a single burner combustion chambers.

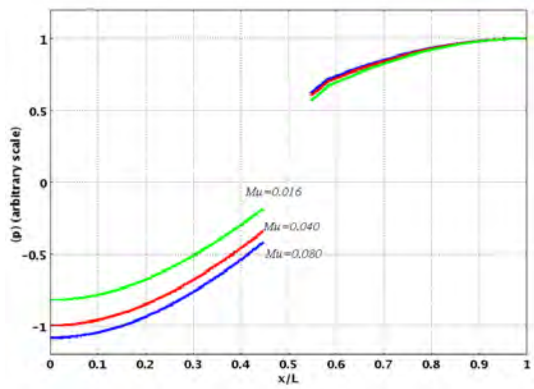

Figure 6: Modeshape corresponding to the first eigenmode of the straight duct with variation of section (Fig. 1) and $T_2/T_1 = 2$, using Eq. (11) with $\zeta = 1.2$ and variation of M_u.

Fig. 6 shows the influence of the Mach number on the first eigenmode.

This test is useful in order to confirm the capability of the code to use correctly the transfer matrix method together with experimental data. In this way it is possible to carry on this procedure both in design stage and in check stage. In design stage one can simulate the process varying the operating parameters in order to obtain the desired specimens, using analytical transfer matrix and experimental information. In check stage one can verify that the solutions adopted in order to avoid thermoacoustic combustion instabilities succeed in their aim, introducing the obtained experimental results or modelling these solutions inside the finite element code.

Moreover, the main parameters of experimental transfer matrix (Eq. (11)), which are l_{eff} and ζ, have been varied in order to understand their influence on frequency and growth rate of the mode of the examined system.

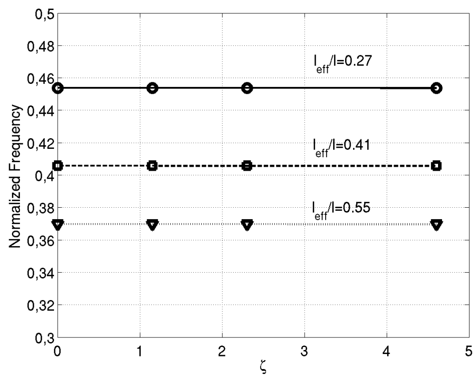

Figure 7: Variation with the loss coefficient of the normalized frequency of the first eigenmode.

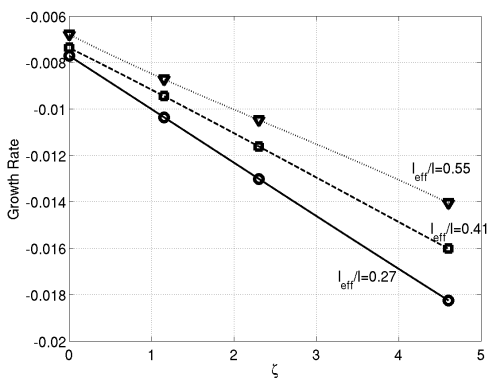

Figure 8: Variation with the loss coefficient of the growth rate of the first eigenmode.

Fig. 7 shows that frequency is influenced only by the effective length of the burner. In fact when the length of the burner increases, consequently the eigenfrequency of the system decreases. On the other hand, Fig. 8 shows that growth rate is influenced by the loss coefficient ζ and the effective length, but the greater effect is due to the loss coefficient. The mode is stable for this configuration, but it becomes more stable when ζ increases, that is when there is a greater damping.

Fig. 9 show the influence of Mach number. Increasing Mach number, frequency decreases such as the growth rate, so that the higher is the velocity of the flow inside the burner the more stable is the eigenmode.

These results show that it is possible to avoid combustion instability designing correctly the shape of the burners and taking into account the geometrical constraints of the combustion chamber. In this kind of analysis a right setting of the fluid dynamic conditions inside the burners have to be well considered, in order to avoid important decrement of static pressure. Introducing Helmholtz resonators, a good match of both fluid dynamic and thermoacoustic operating conditions can be found.

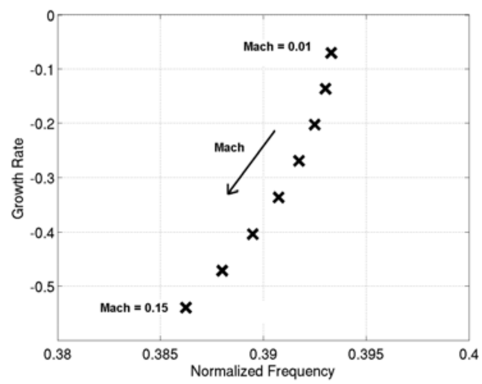

Figure 9: Variation of frequency and growth rate of the first eigenmode for Mach = 0.01 to 0.15 with an increment of 0.02 of the straight duct with variation of section (Fig. 1).

5 Conclusion

An approach, based on the use of a finite element acoustic code, is proposed for the simulation of the combustion instability in the frequency domain introducing the transfer matrix method. In order to establish proof of concept, the present method has been applied to several test cases. Frequencies and growth rate have been accurately caught and results are in a very good agreement with results obtained without the application of transfer matrix method.

This kind of approach is appropriate to treat complex geometries based on real dimensions and shapes, which are really difficult to be treated by means of analytical methods or Acoustic Network methods. In this FEM analysis different kinds of transfer matrix functions and flame transfer functions, both analytical and experimental ones, have been successfully applied. Even in this work, the FEM analysis has been carried out under the hypothesis of negligible mean flow velocity. This is not a strong limitation in the application of this method to real combustion chamber and real operating conditions. In fact, this hypothesis is applied only to plenum and combustion chamber where Mach number is generally far from sonic condition. On the other hand, this hypothesis is not referred to the burners, because burners are substituted by an appropriate transfer matrix function, which links pressure and velocity fluctuations at both ends of such components, taking into account the effect of mean flow on the acoustic wave oscillations.

The main aim of the application of the Transfer Matrix method has been the evaluation of the substitution of a domain with an analytical (on experimental) transfer matrix, in order to reduce the computational efforts when a realistic geometry is studied. The substitution of a piece of geometry can be useful when this element has a complex shape, which requires a very fine computational mesh.

This approach has been applied even to annular geometries, obtaining the same conclusion, that is a very good agreement between the results with and without the apllication of burner transfer matrix [3]. On the more the transfer matrix method has been successfully applied in the acoustic analysis of a realistic gas turbine combustion chamber, substituting the burners with transfer matrices, so considering important data coming from CFD analysis.

In conclusion, the proposed method appears to be a simple tool for the analysis of thermoacoustic combustion instability both in design stage and in check stage, taking the hypothesis of linear acoustic waves. Additionally, this kind of application of the Transfer Matrix method can be extended to other acoustic and not field.

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