Optimal Design for the Grating Coupler of Surface Plasmons

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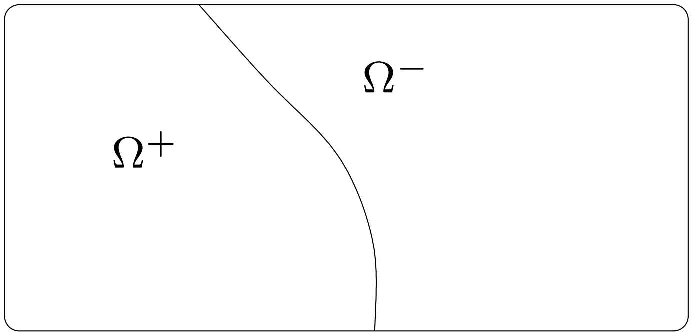
Outline

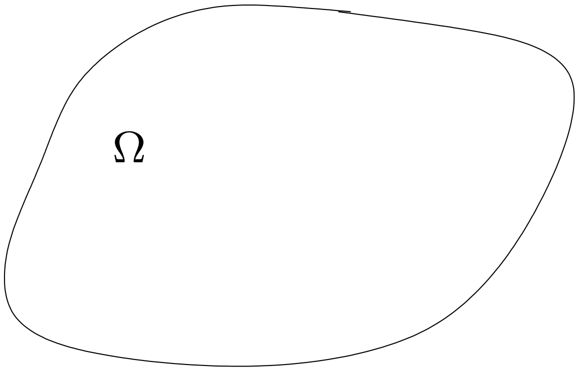
- General formulation of the shape optimization
- Background of Plasmon Wave
- Shape gradient of the grating coupler
- Numerical Results

General Problem Formulation

$$\min_{\Omega \in \mathcal{A}_{ad}} J(\Omega) = j(\Phi(\Omega))$$
$$A(\Phi(\Omega)) = F$$

where A is a differential operator.





Adjoint Variable(Lagrange Multiplier)

$$L(\Phi, \Psi) = j(\Phi) - \int_{\Omega} (A(\Phi) - F)\Psi dx$$

Taking the variable w.r.t Φ, we can get the equation for Ψ

$$\delta_{\Phi} L = 0$$

or

$$A^{*}(\Psi) = \partial_{\Phi} j(\Phi)$$

Dynamic Shape Optimization(1)

Let Ω evolves with time, i.e

$$\frac{d}{dt}x = \mathbf{u}(x,t), \quad x \in \Omega_t$$

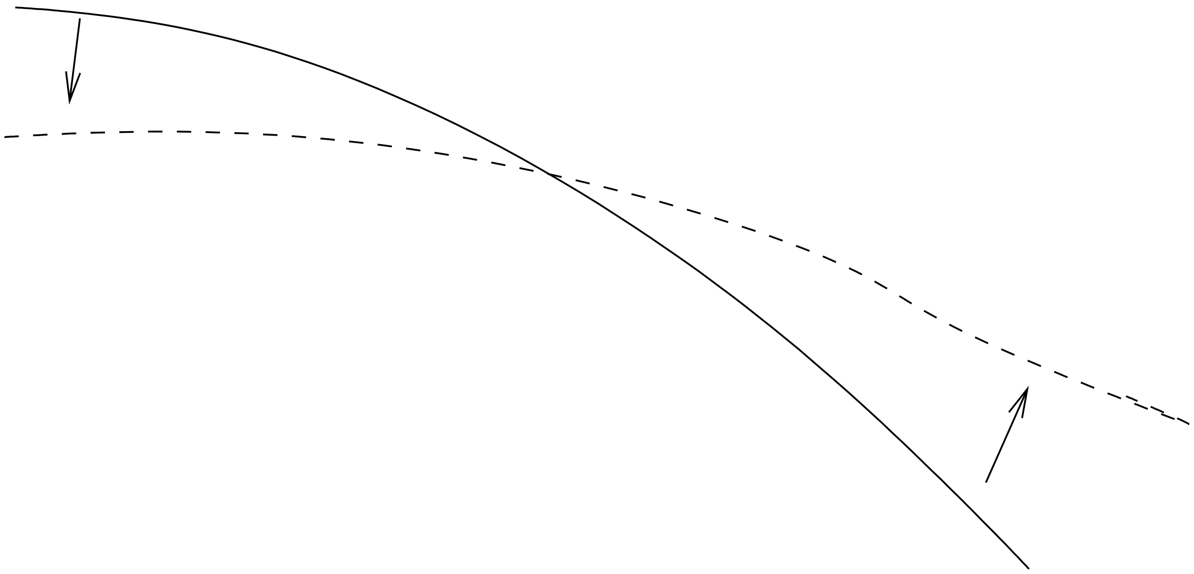
Then

$$\begin{aligned} \frac{d}{dt}j(\Phi) &= \langle \partial_\Phi j(\Phi), \frac{d}{dt}\Phi \rangle \\ &= \langle A^*(\Psi), \frac{d}{dt}\Phi \rangle \end{aligned}$$

Dynamic Shape Optimization(2)

$$
\begin{aligned}
\frac{d}{dt}j(\Phi) &= \langle \partial_\Phi j(\Phi), \frac{d}{dt}\Phi \rangle \\
&= \langle A^*(\Psi), \frac{d}{dt}\Phi \rangle \\
&= \langle \Psi, A\frac{d}{dt}\Phi \rangle \\
&= \langle \Psi, \frac{d}{dt}(A\Phi) \rangle - \langle \Psi, (\frac{d}{dt}A)\Phi \rangle \\
&= -\langle \Psi, (\frac{d}{dt}A)\Phi \rangle
\end{aligned}
$$

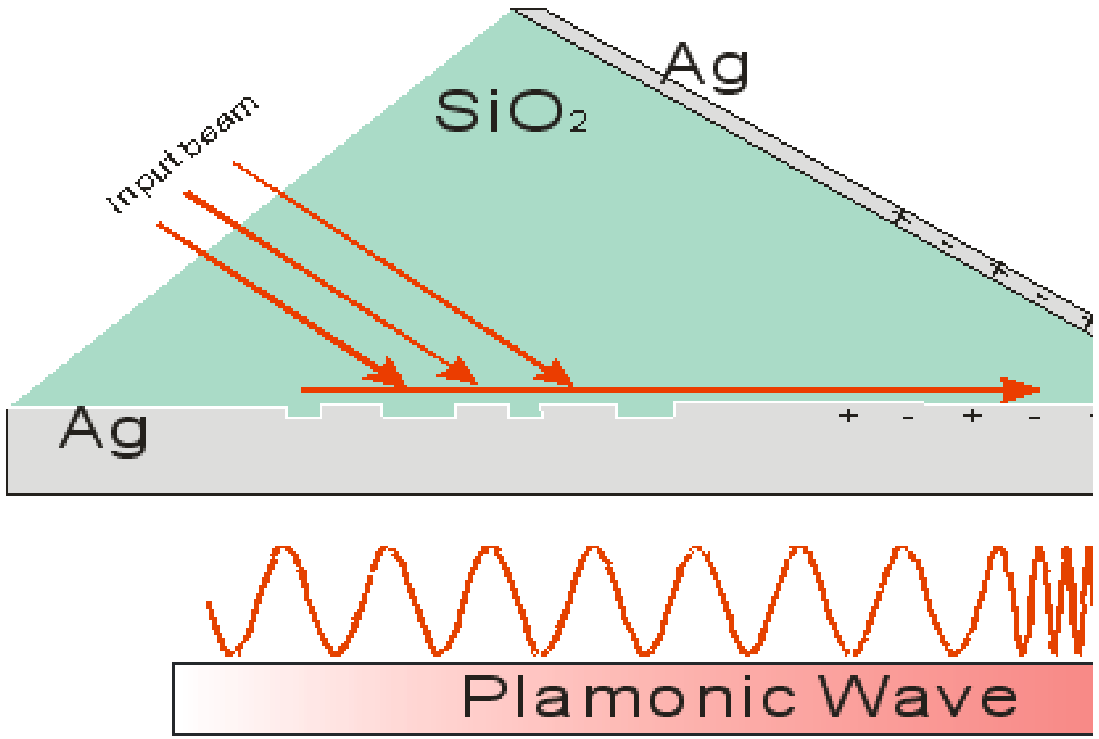
Dynamic Shape Optimization(3)



Usually,

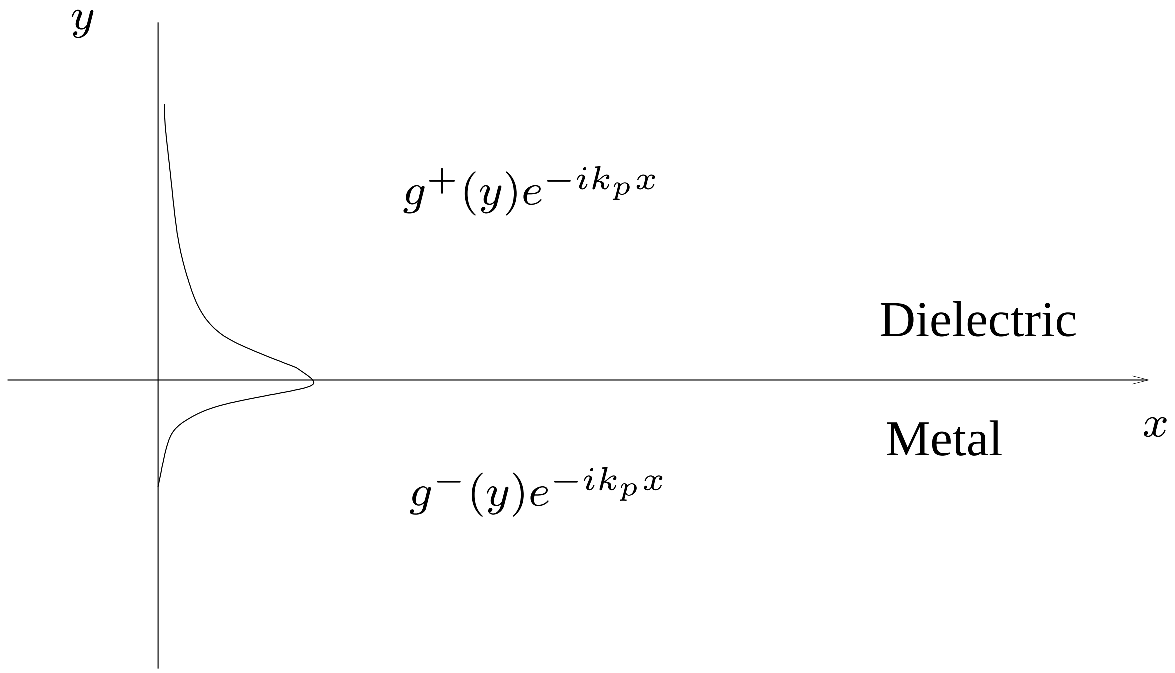
$$\frac{d}{dt} j(\Phi) = \int_{\Gamma_m} f(\Phi, \Psi) \mathbf{u} \cdot \mathbf{n} dS$$

Physical Background of Plasmon Wave



Physical Background of Plasmon Wave

A Wave travelling along the interface between metal and dielectrics.



Physical Background of Plasmon Wave

The dispersion relation is given by

$$k_p = \sqrt{\frac{\epsilon_m(\omega)\epsilon_d(\omega)}{\epsilon_m(\omega) + \epsilon_d(\omega)}}$$

For existence of plasmon wave

$$\epsilon_m(\omega) < -\epsilon_d(\omega)$$

Physical Background of Plasmon Wave

$$\nabla \times \mathbf{H} = \frac{\partial(\epsilon \mathbf{E})}{\partial t}$$

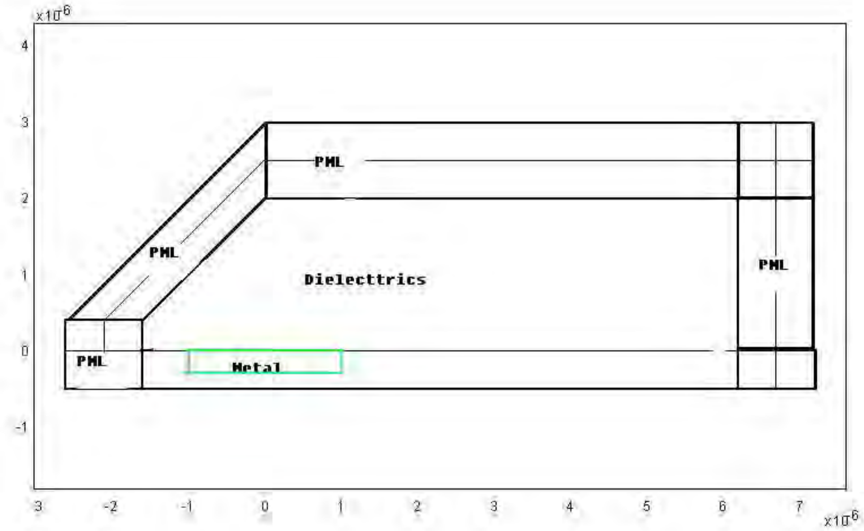
$$\nabla \times \mathbf{E} = -\frac{\partial(\mu \mathbf{H})}{\partial t}$$

Time Harmonic, Transverse Magnetic

$$\mathbf{E} = \begin{pmatrix} E_x(x,y) \\ E_y(x,y) \\ 0 \end{pmatrix} e^{j\omega t}, \quad \mathbf{H} = \begin{pmatrix} 0 \\ 0 \\ H_z(x,y) \end{pmatrix} e^{j\omega t}$$

Physical Background of Plasmon Wave

$$\nabla \cdot \epsilon_r^{-1} \nabla H_z + k_0^2 H_z = 0, \quad k_0^2 = \omega^2 \epsilon_0 \mu_0$$



Problem Formulation

$$\min J = \frac{1}{2}\int_\Gamma Re(\vec{E} \times \vec{H}^*) \cdot \mathbf{n} ds$$

Using the relation

$$\begin{aligned}
\vec{E} = \frac{1}{j\omega\epsilon}\nabla \times \vec{H} &= \frac{1}{j\omega\epsilon}(\frac{\partial Hz}{\partial y}, -\frac{\partial Hz}{\partial x}, 0) \\
Re(\vec{E} \times \vec{H}) \cdot \mathbf{n} &= Re(\frac{1}{j\omega\epsilon}(-\frac{\partial Hz}{\partial x}Hz^*, \frac{\partial Hz}{\partial y}Hz^*, 0)) \cdot (1,0,0) \\
&= Re(\frac{j}{\omega\epsilon}\frac{\partial Hz}{\partial x}Hz^*) \\
\frac{\partial Hz}{\partial x} &= -jk_p Hz
\end{aligned}$$

Problem Formulation

$$Re(\vec{E} \times \vec{H}^*) \cdot \mathbf{n} = \frac{1}{\omega Re(\epsilon)} |Hz|^2$$

min

$$J = \frac{1}{2} \int_\Gamma \frac{1}{Re(\epsilon)} |Hz|^2 dx$$

$$\nabla \cdot \epsilon_r^{-1} \nabla H_z + k_0^2 H_z = 0$$

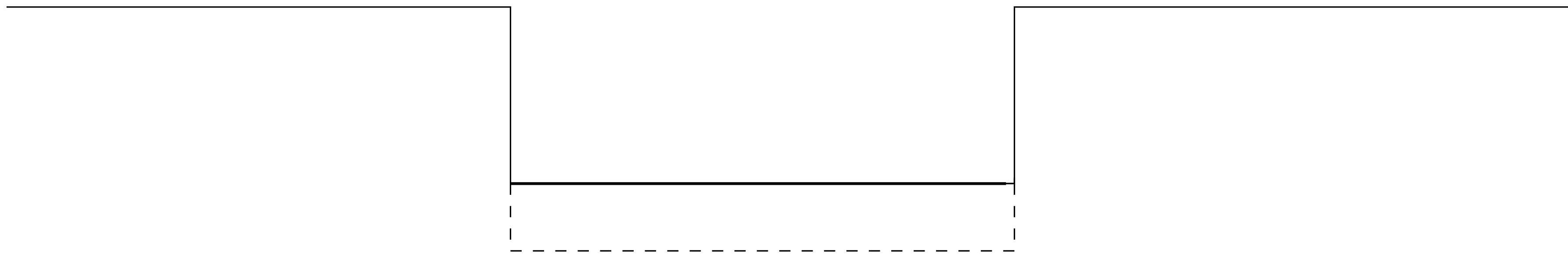
$$\nabla \cdot \epsilon_r^{-1} \nabla \tilde{H}_z + k_0^2 \tilde{H}_z = \frac{1}{Re(\epsilon_r)} Hz^* \delta_\Gamma$$

Gradient of the Objective Functional w.r.t time

$$
\begin{aligned}
\frac{d}{dt}J &= Re\left(\int_{\gamma}\left[\frac{1}{\epsilon_r}(2\frac{\partial Hz}{\partial n}\frac{\partial \tilde{H}z}{\partial n} - \nabla Hz \cdot \nabla \tilde{H}z)\right] ds\right) \\
&= Re\left((\frac{1}{\epsilon_r^+} - \frac{1}{\epsilon_r^-})\int_{\gamma} \nabla Hz^+ \cdot \nabla \tilde{H}z^- \mathbf{u}\cdot\mathbf{n} ds\right) \\
&= Re\left((\frac{1}{\epsilon_r^+} - \frac{1}{\epsilon_r^-})\int_{\gamma} \nabla Hz^- \cdot \nabla \tilde{H}z^+ \mathbf{u}\cdot\mathbf{n} ds\right)
\end{aligned}
$$

Gradient of the Objective Functional w.r.t position

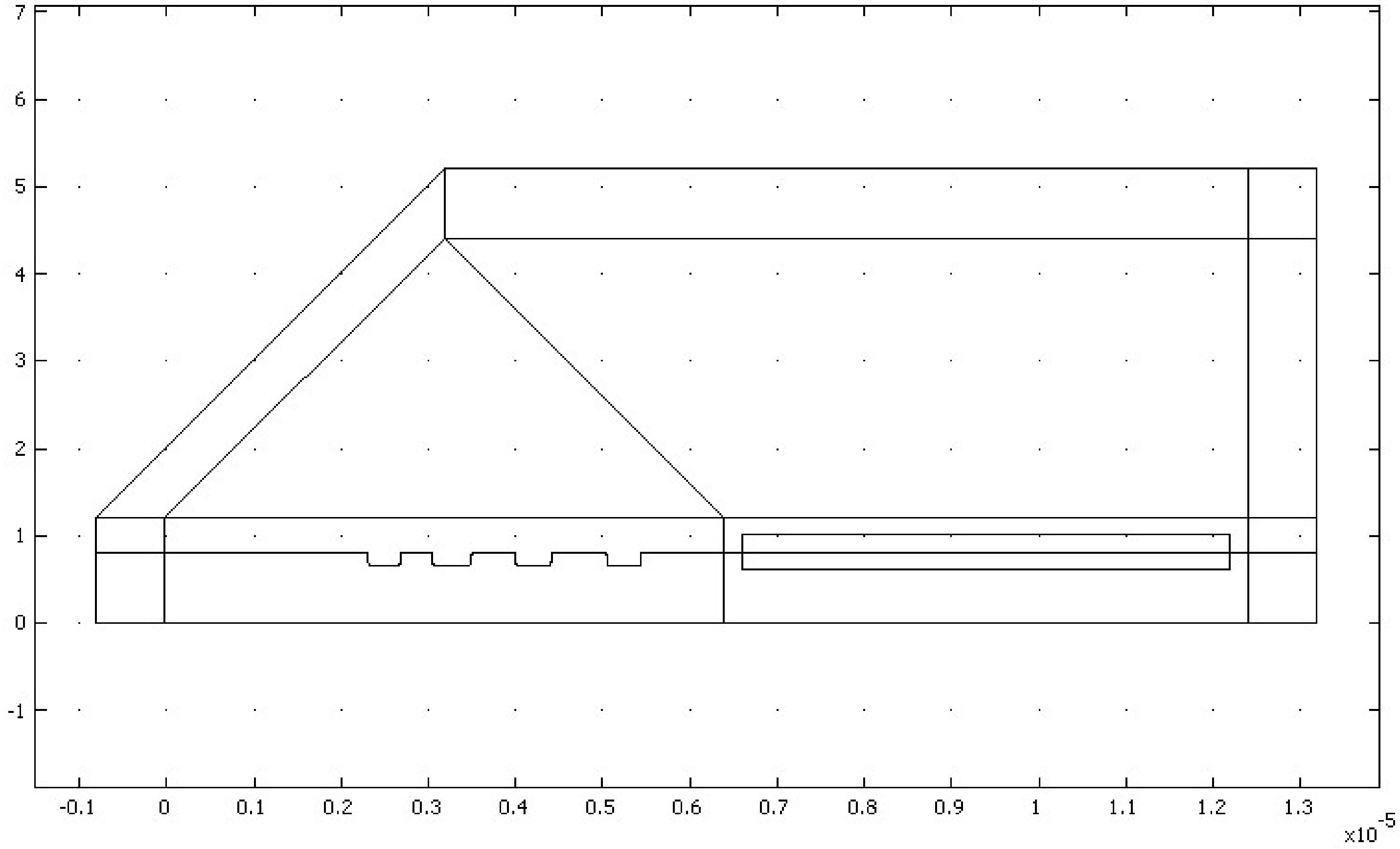
For finite number of gratings,
$J = J(x_1, x_2, \cdots, x_N, h)$,



$$\frac{dJ}{dt} = \frac{\partial J}{\partial h} v_h = \int_{\gamma_h} \nabla Hz^{-} \cdot \nabla \tilde{H}z^{+} v_h ds$$

$$\frac{\partial J}{\partial h} = \int_{\gamma_h} \nabla Hz^{-} \cdot \nabla \tilde{H}z^{+} ds$$

Geometry

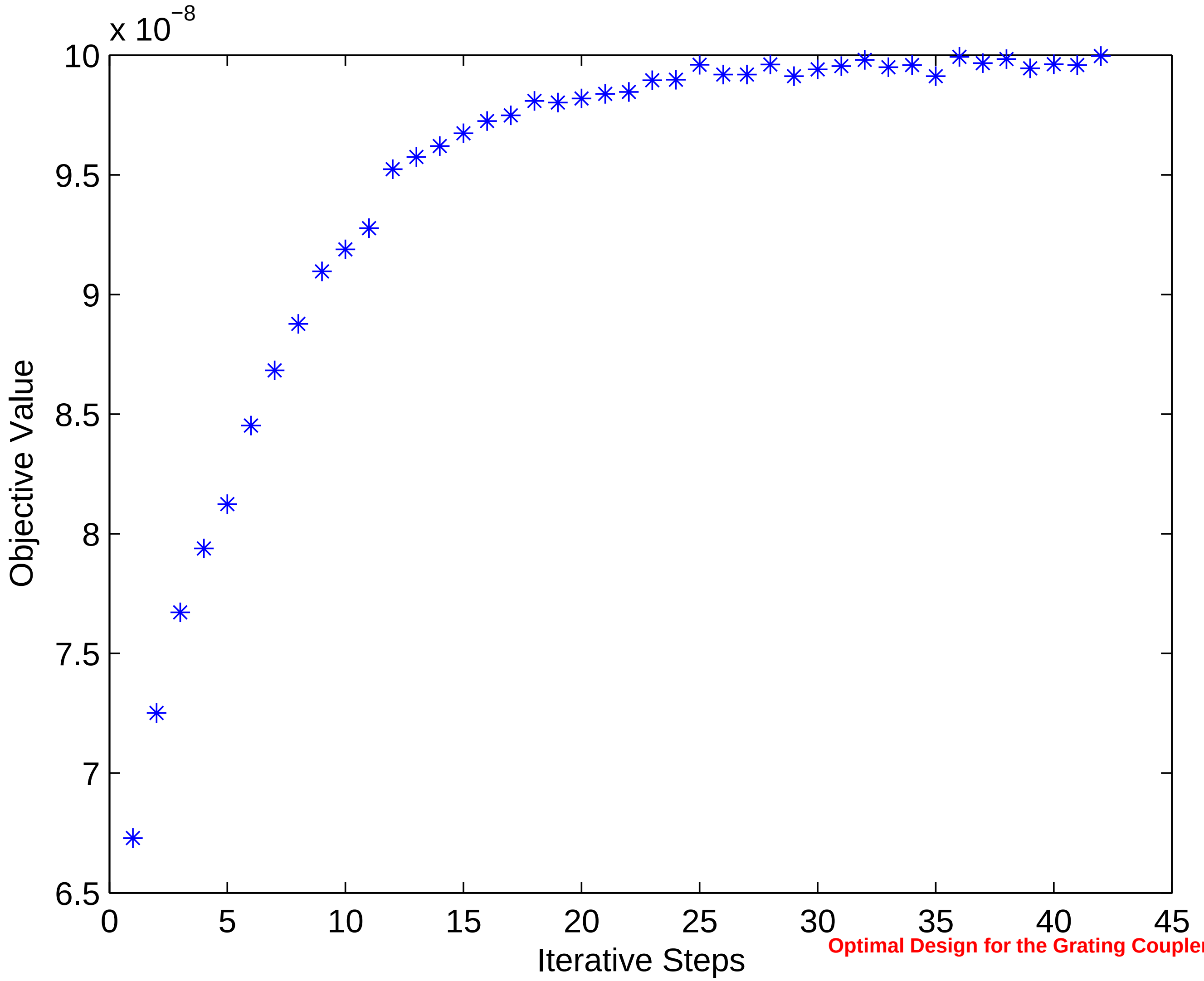


Numerical algorithm

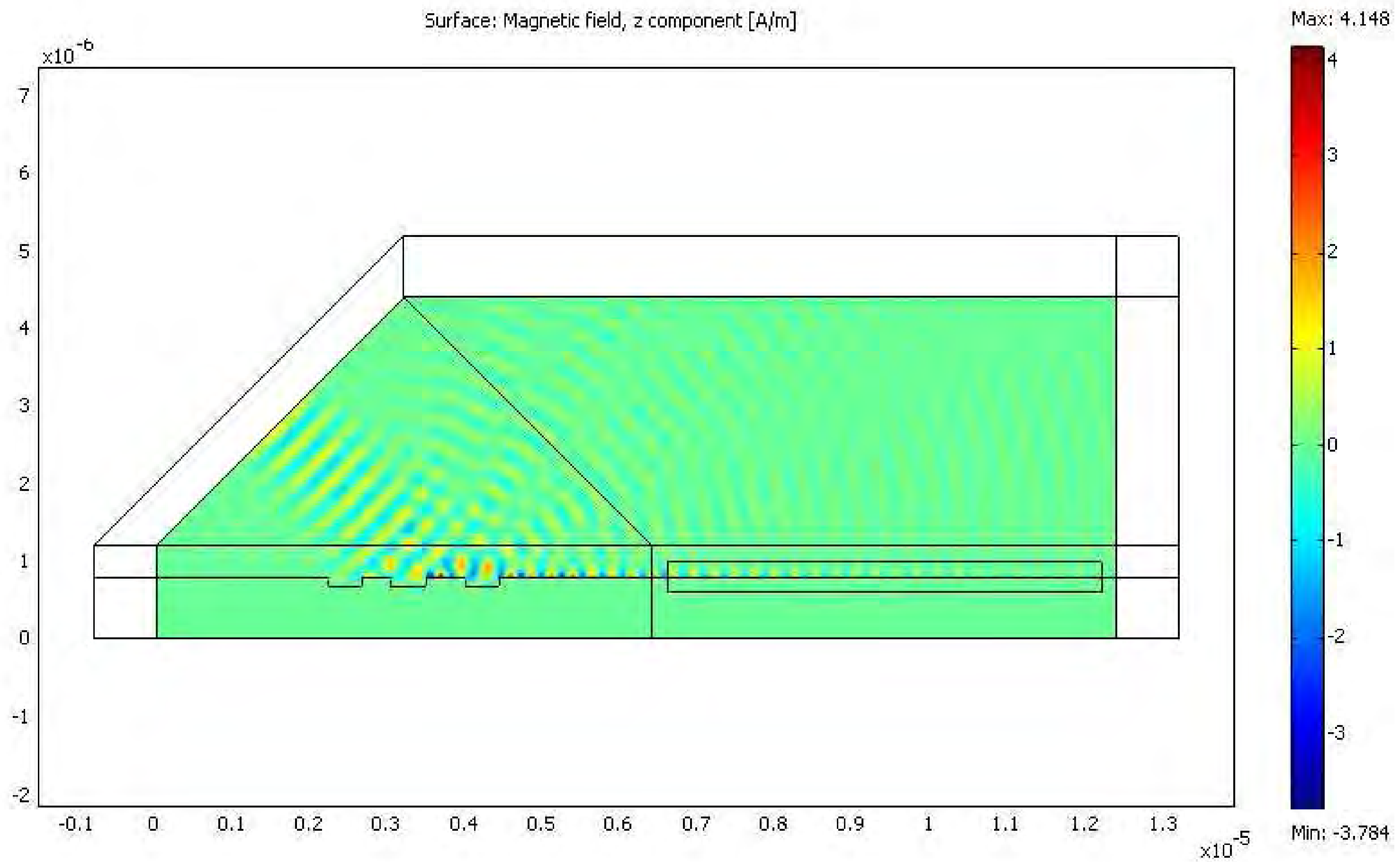
During each iteration

0. Solve $\nabla \cdot \frac{1}{\epsilon}\nabla H_z + k_0^2 H_z = 0$
0. Solve for the adjoint variable
   $\nabla \cdot \epsilon_r^{-1}\nabla \tilde{\tilde{H}}_z + k_0^2 \tilde{\tilde{H}}_z = \frac{1}{Re(\epsilon_r)} Hz^* \delta_\Gamma$
0. Calculate the partial derivatives $\frac{\partial J}{\partial x_i}, \frac{\partial J}{\partial h}$
0. Gradient ascent to update the geometry

Numerical Result



Numerical Result



The End!