FEM Convergence for PDEs with Point Sources in 2-D and 3-D

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Abstract

The finite element method (FEM) is widely used as a numerical method for the solution of partial differential equation (PDE) problems, especially for elliptic PDEs such as the Poisson equation with Dirichlet boundary conditions. We consider this problem here on both two- and three-dimensional domains. The FEM solution will typically incur an error against the PDE solution of the problem.

Convergence analysis results for the FEM error necessitate requirements on the finite element method used as well as on the PDE problem [1]:
(i) Lagrange finite elements, such as those in COMSOL Multiphysics® software, approximate the PDE solution at several points in each element of a mesh, such that the restriction of the FEM solution to each element is a polynomial of a chosen degree and the FEM solution is continuous across all boundaries between neighboring elements throughout the domain. For the case of linear Lagrange elements with piecewise linear polynomial degree, the FEM convergence is quadratic, i.e., one higher than the polynomial degree [1]; it also holds under additional assumptions on the PDE problem that for higher-order elements, the convergence order can reach one order higher than the polynomial degree.
(ii) One necessary assumption on the PDE is that the problem has a solution that is sufficiently regular, as expressed by the number of derivatives that it has. In the context of the FEM, it is appropriate to consider weak derivatives [1].
The convergence order of the FEM with Lagrange elements is then limited by the regularity order of the PDE solution. The facts (i) and (ii) can be combined to predict the convergence order. For linear Lagrange elements for instance, we need the PDE solution to have two orders of weak derivatives in order to reach the optimal quadratic convergence order.

The purpose of this paper is to show how one can demonstrate computationally the quadratic convergence order for linear Lagrange FEM elements, if the PDE solution is smooth enough. We then demonstrate that the convergence order is indeed limited by the regularity of the PDE solution. This latter situation arises concretely when considering a PDE with one (or more) point sources in the forcing term on the right-hand side of the PDE. The reason is that the mathematical model for point sources is given by the Dirac delta distribution. This function is highly non-

smooth, and thus the PDE solution does not have two orders of derivatives. Specifically, the computational results show that the convergence order for non-smooth forcing is 1 for two-dimensional problems and 0.5 for three-dimensional problems, consistent with available theory [2, 3]. This demonstrates that PDE problems with a non-smooth source term necessarily have degraded convergence order compared to problems with smooth right-hand sides and thus can be most efficiently solved by low-order FEM such as linear Lagrange elements. In [4], we provide detailed instructions for obtaining the results of this report in COMSOL 5.1.

Reference

[1] Dietrich Braess. Finite Elements. Cambridge University Press, third edition (2007).

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[3] Thomas I. Seidman et al., Finite Element Approximation for Time-Dependent Diffusion with Measure-Valued Source, Numer. Math., vol. 122, no. 4, pp. 709-723 (2012).

[4] Kourosh M. Kalayeh et al., FEM Convergence Studies for 2-D and 3-D Elliptic PDEs with Smooth and Non-Smooth Source Terms in COMSOL 5.1, Technical Report HPCF-2015-19, UMBC High Performance Computing Facility, University of Maryland, Baltimore County (2015).

Figures used in the abstract

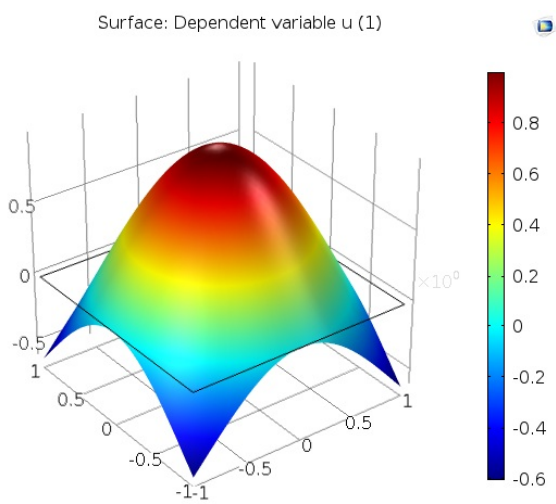

Figure 1: Three-dimensional view of the FEM solution for the smooth test problem.

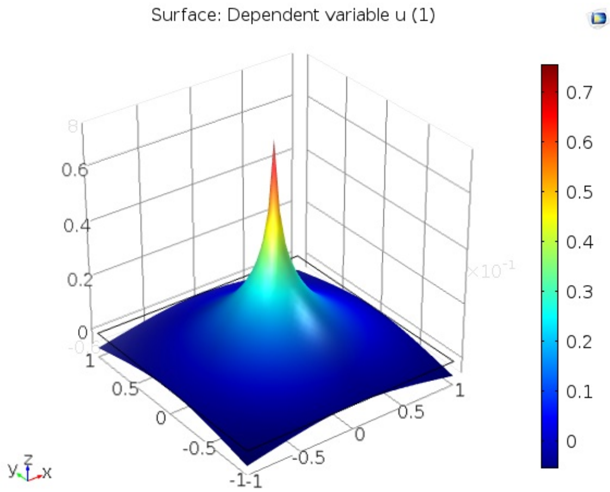

Figure 3: Three-dimensional view of the FEM solution for the non-smooth test problem