Edge Element and Second-Order Nodal Analysis for Arbitrary Shaped Waveguides

H. Arab[*1], F. Afshar[2], and C. Akyel[1]
[1]Ecole Polytechnique of Montreal, [2]Mcgill University
*Corresponding author: 5120 Earnscliffe, Montreal, QC, Canada, arab-salmanabadi.homa@polymtl.ca

Abstract: In this project a two dimensional second order nodal and linear edge elements programming model for homogeneous waveguide is developed and simulated in Matlab software Environment. The objective is to reduce or eliminate spurious solutions and to cater for any arbitrarily shaped waveguides using triangular edge elements. The formulation is developed using the E-Field and to make use of the edge variables in the Transversal-Plane and the node variables in the Longitudinal-plane. The validity and effectiveness of the proposed method are investigated and compared with the commercial software Comsol Multiphysics and analytical solution found in literature. The proposed method is capable of dealing with multitude of configurations with reduced computational complexity and time due to the considerable reduction in mesh size and shape function order.

Keywords: Waveguide, Numerical Method, Finite element, Edge Element

1. Introduction

As a result of the vast variety of practical applications of the dielectric filled waveguide in microwave and optical frequencies, the development of methods to solve the associated electromagnetic field problems has attracted the attention of many researchers. The finite-element method, to compute accurately the behavior of the waveguide with arbitrary cross section, has been widely used [1]–[7]. Nodal based calculations encounter some short comes such as appearance of spurious solutions, inconvenience of imposing boundary conditions at material interfaces and difficulty in treating conducting and dielectric edges and corners due to the field singularities [8]. Avoiding all these shortcomings redirected attentions toward new approaches, beyond simple modifications, leading to a new era called vector basis or edge elements. Edge element wave analysis is essential to evaluate the propagation characteristics of microwave and optical waveguides with arbitrarily shaped cross sections. Constant edge elements for transverse components of the electric or magnetic field and linear nodal elements for the axial one has been previously used for solving directly the propagation constant [9]-[11]. This approach provides a direct solution for calculating the propagation constants [10] avoiding spurious solutions. However, the accuracy of the finite element analysis using the lowest order elements is, in general, insufficient [12]. In this study, to enhance the accuracy and faster convergence, a vector finite element method with the second-order nodal elements is formulated in details. It is a combination of linear edge elements for transverse components of the electric field and quadratic nodal elements for the axial one. Furthermore, the accuracy of this approach is investigated by calculating the propagation constant of waveguide with numerical examples.

2. Basic Equations

Consider an arbitrarily shaped metal dielectric. This waveguide is assumed to be uniform along its longitudinal z axis. Maxwell curl equations for time-harmonic fields are:

$$\nabla \times E = -j\omega\mu H \qquad (1)$$
$$\nabla \times H = j\omega\epsilon E \qquad (2)$$

where the vectors E and H are the electric and the magnetic-field intensities, respectively. ϵ is permittivity and μ denote the permeability, which are assumed to be constant in waveguide. ω is an angular frequency. From (1) and (2), we construct:

$$E = -\left(\frac{j}{\omega}\right)\epsilon^{-1}\nabla \times H \qquad (3)$$
$$H = \left(\frac{j}{\omega}\right)\mu^{-1}\nabla \times E \qquad (4)$$

By taking the curl of (3) and (4), and then substituting into (1) and (2), the following common curl-curl equation for E is obtained:

$$\nabla \times \left(\frac{1}{\mu}\nabla \times E\right) - k_0^2\epsilon E = 0 \qquad in \quad \Omega \qquad (5)$$

where k_0 is the free-space wave number and Ω is the waveguide cross section. An appropriate boundary condition must be satisfied by the field vectors:

$$n \times E = 0 \qquad on\ \Gamma 1 \qquad (6)$$
$$n \times (\nabla \times E) = 0 \qquad on\ \Gamma 2 \qquad (7)$$

where n is a surface normal unit vector. $\Gamma 1$ is electric wall and $\Gamma 2$ is the magnetic wall.

The variational principal for this problem is given by:

$$\partial F(E) = 0 \tag{8}$$
$$n \times E = 0 \qquad on\ \Gamma 1 \tag{9}$$

where

$$F(E) = \frac{1}{2}\iint_{\Omega}\left[\frac{1}{\mu}(\nabla \times E).(\nabla \times E)^* - k_0^2\epsilon E.E^*\right] d\,\Omega \tag{10}$$

By assuming z is the direction of propagation we will have $E(x,y,z) = E(x,y)e^{-jk_z z}$ where k_z is the propagation constant, so (10) can be written as:

$$F(E) = \frac{1}{2}\iint_{\Omega} \frac{1}{\mu}(\nabla_t \times E_t).(\nabla_t \times E_t)^* - k_0^2\epsilon E.E^* d\,\Omega + \frac{1}{\mu}(\nabla_t E_z + jk_z E_t).(\nabla_t E_z + jk_z E_t)^* d\Omega \tag{11}$$

where ∇_t denotes the transverse del operator, E_t denotes the transverse component of the electric field, and E_z the z-component of the field. This functional can be discretized to yield an eigenvalue system that can be solved for k_0^2 for a given k_z [8].

3. Finite element formulations

The electromagnetic fields have to be tangentially continuous across material interfaces. In the edge element [13]-[20]**,** the tangential continuity can be straightforwardly imposed.

3.1 Triangular Edge Elements

The cross section of the waveguide is subdivided into a finite number of triangular elements. For the second order nodal configuration there is one node in between each corner nodes. The node number and the corresponding node coordinate at each vertex are assigned as *1*, *(x1, y1)*, *2*, *(x2, y2)*, and for *3*, *(x3, y3)*, respectively. The direction of edges is considered from lower value node to the higher one as shown in Fig. 2. The corner points 1 to 3 are for the axial component E_z, while the side points 4 to 6 are for the tangential component of E_z. The axial component E_z is approximated by a complete polynomial of second order.

3.2 Finite-Element Discretization

For solving eq. (11) Lee et al [21] introduced the following transformation of variables:

$$e_t = k_z E_{t,} \qquad e_z = -jE_{z,} \tag{12}$$

Substituting eq. (12) into eq. (11) and multiplying it by k_0^2 resulted into:

$$F(e) = \frac{1}{2}\iint_{\Omega} \frac{1}{\mu}(\nabla_t \times e_t).(\nabla_t \times e_t)^* - k_0^2\epsilon e_t.e_t{}^* d + k_z^2\left[\frac{1}{\mu}(\nabla_t e_z + e_t).(\nabla_t e_z + e_t)^* - k_0^2\epsilon e_t.e_t{}^*\right] d\Omega \tag{13}$$

By subdividing the cross sectional area Ω into small triangular elements, M elements, within each element the vector transverse field e_t and the longitudinal component e_z can be expanded using the conventional node-based interpolation functions:

$$e_t^e = \sum_{i=1}^{3} N_i^e e_{ti}^e = \{N^e\}^T\{e_t^e\} = \{e_t^e\}^T\{N^e\} \tag{14}$$
$$e_z^e = \sum_{i=1}^{3} N_i^e e_{zi}^e = \{N^e\}^T\{e_z^e\} = \{e_z^e\}^T\{N^e\} \tag{15}$$

By substituting these into eq. (13) one obtains:

$$F(e) = \frac{1}{2}\sum_{e=1}^{M}\left(\{e_t^e\}^T[A_{tt}^e]\{e_t^{e*}\} + k_z^2\begin{bmatrix} B_{tt}^e & B_{tz}^e \\ B_{zt}^e & B_{zz}^e\end{bmatrix}\begin{Bmatrix} e_t^e \\ e_z^e\end{Bmatrix}^*\right) \tag{16}$$

where the elemental matrices are given by

$$[A_{tt}^e] = \iint_{\Omega^e}\left[\frac{1}{\mu}\{\nabla_t \times \mathbf{N}^e\}.\{\nabla_t \times \mathbf{N}^e\}^T - k_0^2\epsilon\{\mathbf{N}^e\}.\{\mathbf{N}^e\}^T\right] d\Omega \tag{17}$$
$$[B_{tt}^e] = \iint_{\Omega^e}\frac{1}{\mu}\{\mathbf{N^e}\}.\{\mathbf{N^e}\}^T d\,\Omega \tag{18}$$
$$[B_{tz}^e] = \iint_{\Omega^e}\frac{1}{\mu}\{\boldsymbol{N}^e\}.\{\nabla_t N^e\}^T d\,\Omega \tag{19}$$
$$[B_{zt}^e] = \iint_{\Omega^e}\frac{1}{\mu}\{\nabla_t N^e\}.\{\boldsymbol{N}^e\}^T d\,\Omega \tag{20}$$
$$[B_{zz}^e] = \iint_{\Omega^e}\left[\frac{1}{\mu}\{\nabla_t N^e\}.\{\nabla_t N^e\}^T - k_0^2\epsilon\{N^e\}.\{N^e\}^T\right] d\,\Omega \tag{21}$$

where Ω^e is the area of the elements. Both permittivity and permeability are constant within the element and can be evaluated analytically. $[A_{tt}^e]$ and $[B_{tt}^e]$ formulations for second order elements are [8]:

$$A_{ij}^e = \frac{4\delta_{ij}-1}{12\Delta^e}\left(a_x^e b_i^e b_j^e + a_y^e c_i^e c_j^e\right) \qquad i,j = 1,2,3.$$
$$A_{14}^e = A_{24}^e = -4A_{12}^e \qquad A_{16}^e = A_{36}^e = -4A_{13}^e$$
$$A_{25}^e = A_{35}^e = -4A_{23}^e\,, \qquad A_{15}^e = A_{26}^e = -4A_{34}^e = 0$$
$$A_{44}^e = \frac{2}{3\Delta^e}\left[a_x^e(b_1^{e2} + b_1^e b_2^e + b_2^{e2}) + a_y^e(c_1^{e2} + c_1^e c_2^e + c_2^{e2})\right]$$
$$A_{55}^e = \frac{2}{3\Delta^e}\left[a_x^e(b_3^{e2} + b_1^e b_3^e + b_1^{e2}) + a_y^e(c_3^{e2} + c_1^e c_3^e + c_1^{e2})\right]$$
$$A_{66}^e = \frac{2}{3\Delta^e}\left[a_x^e(b_1^{e2} + b_1^e b_2^e + b_2^{e2}) + a_y^e(c_1^{e2} + c_1^e c_2^e + c_2^{e2})\right]$$
$$A_{45}^e = \frac{1}{3\Delta^e}\left[a_x^e(b_3^e b_2^e + 2b_1^e b_3^e + b_1^e b_2^e + b_2^{e2}) + a_y^e(c_3^e c_2^e + 2c_1^e c_3^e + c_1^e c_2^e + c_2^{e2})\right]$$
$$A_{45}^e = \frac{1}{3\Delta^e}\left[a_x^e(b_3^e b_2^e + 2b_1^e b_3^e + b_1^e b_2^e + b_2^{e2}) + a_y^e(c_3^e c_2^e + 2c_1^e c_3^e + c_1^e c_2^e + c_2^{e2})\right]$$
$$A_{46}^e = \frac{1}{3\Delta^e}\left[a_x^e(b_3^e b_1^e + 2b_2^e b_3^e + b_1^e b_2^e + b_1^{e2}) + a_y^e(c_3^e c_1^e + 2c_2^e c_3^e + c_1^e c_2^e + c_1^{e2})\right]$$
$$A_{56}^e = \frac{1}{3\Delta^e}\left[a_x^e(b_3^e b_1^e + 2b_1^e b_2^e + b_3^e b_2^e + b_3^{e2}) + a_y^e(c_3^e c_1^e + 2c_1^e c_2^e + c_3^e c_2^e + c_3^{e2})\right]$$

$$[B^e] = \frac{\beta\Delta^e}{180}\begin{bmatrix} 6 & -1 & -1 & 0 & -4 & 0 \\ -1 & 6 & -1 & 0 & 0 & -4 \\ -1 & -1 & 6 & -4 & 0 & 0 \\ 0 & 0 & -4 & 32 & 16 & 16 \\ -4 & 0 & 0 & 16 & 32 & 16 \\ 0 & -4 & 0 & 16 & 16 & 32 \end{bmatrix}$$

By applying Ritz procedure the generalized eigenvalue problem is obtained:

$$\begin{bmatrix} A_{tt}^e & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} e_t \\ e_z \end{Bmatrix} = -k_z^2 \begin{bmatrix} B_{tt}^e & B_{tz}^e \\ B_{zt}^e & B_{zz}^e \end{bmatrix} \begin{Bmatrix} e_t^e \\ e_z^e \end{Bmatrix}^* \qquad (22)$$

By applying second order formulation for triangular elements, the interpolation functions will be:

$$N_j^e(x,y) = \left(2L_j^e - 1\right)L_j^e, \qquad j = 1,2,3 \qquad (23.a)$$

$$N_4^e(x,y) = 4L_1^e L_2^e, \;\; N_5^e(x,y) = 4L_2^e L_3^e,$$
$$N_6^e(x,y) = 4L_3^e L_1^e \qquad (23.b)$$

In the above, L_j^e are given by

$$L_j^e(x,y) = \frac{1}{2\Delta^e}\left(a_j^e + b_j^e + c_j^e\right) \qquad j = 1,2,3 \qquad (24)$$

where a_j^e, b_j^e, c_j^e *and* Δ^e are:

$$a_1^e = x_2^e y_3^e - y_2^e x_3^e,\;\; b_1^e = y_2^e - y_3^e,\;\; c_1^e = x_3^e - x_2^e \qquad (25.a)$$
$$a_2^e = x_3^e y_1^e - y_3^e x_1^e,\;\; b_2^e = y_3^e - y_1^e,\;\; c_2^e = x_1^e - x_3^e \qquad (25.b)$$
$$a_3^e = x_1^e y_2^e - y_1^e x_2^e,\;\; b_3^e = y_1^e - y_2^e,\;\; c_3^e = x_2^e - x_1^e \qquad (25.c)$$
$$\Delta^e = \frac{1}{2}\left(b_1^e c_2^e - b_2^e c_1^e\right) \qquad (25.d)$$

In the edge element basis function formulation, the reference direction is usually based on the global node numbers at the end point of the corresponding edge. One of the problems which may occur in one or several basis function of an element that share an edge with another one is the reverse direction of the common edge with respect to each other. The solution for this problem is to sort the nodes of all elements in ascending order so the basis function will be defined as below:

$$\mathbf{N}_1^e = \mathrm{w}_{12}\mathrm{l}_1 = (\mathrm{L}_1\nabla\mathrm{L}_2 - \mathrm{L}_2\nabla\mathrm{L}_1)\mathrm{l}_1 \qquad (26.a)$$
$$\mathbf{N}_2^e = \mathrm{w}_{13}\mathrm{l}_2 = (\mathrm{L}_1\nabla\mathrm{L}_3 - \mathrm{L}_3\nabla\mathrm{L}_1)\mathrm{l}_2 \qquad (26.b)$$
$$\mathbf{N}_3^e = \mathrm{w}_{23}\mathrm{l}_3 = (\mathrm{L}_2\nabla\mathrm{L}_3 - \mathrm{L}_3\nabla\mathrm{L}_2)\mathrm{l}_3 \qquad (26.c)$$

By considering following integrals of a vector wave equations [8]:

$$E_{ij}^e = \iint_{\Omega^e} (\nabla \times \mathbf{N}_\mathrm{i}^e).\left(\nabla \times \mathbf{N}_\mathrm{j}^e\right) d\Omega \qquad (27.a)$$
$$F_{ij}^e = \iint_{\Omega^e} \mathbf{N}_\mathrm{i}^e.\mathbf{N}_\mathrm{j}^e \, d\,\Omega \qquad (27.b)$$

We will have:

$$[E^e] = \begin{bmatrix} \frac{l_1 l_1}{\Delta^e} & -\frac{l_1 l_2}{\Delta^e} & \frac{l_1 l_3}{\Delta^e} \\ -\frac{l_2 l_1}{\Delta^e} & \frac{l_2 l_2}{\Delta^e} & -\frac{l_2 l_3}{\Delta^e} \\ \frac{l_3 l_1}{\Delta^e} & -\frac{l_3 l_2}{\Delta^e} & \frac{l_3 l_3}{\Delta^e} \end{bmatrix}$$

$$F_{11}^e = \frac{l_1^{e2}}{24\Delta^e}(f_{33} - f_{13} + f_{11})$$

$$F_{13}^e = -\frac{l_1^e l_2^e}{48\Delta^e}(f_{23} - f_{22} - 2f_{12} + f_{13})$$
$$F_{12}^e = \frac{l_1^e l_3^e}{48\Delta^e}(f_{31} - 2f_{23} - f_{11} + f_{12})$$
$$F_{33}^e = \frac{l_2^{e2}}{24\Delta^e}(f_{22} - f_{23} + f_{33})$$
$$F_{23}^e = -\frac{l_2^e l_3^e}{48\Delta^e}(f_{21} - f_{22} - 2f_{31} + f_{23})$$
$$F_{22}^e = \frac{l_3^{e2}}{24\Delta^e}(f_{11} - f_{12} + f_{22})$$

By using eq.23 and eq24 and eq.27 along with vector interpolation function $[B_{tz}^e]$ matrix could be obtained as follow:

$$B_{11}^e = \frac{l_1}{12\Delta^e}f_{12}, \quad B_{12}^e = B_{13}^e = 0$$
$$B_{21}^e = -\frac{l_1}{12\Delta^e}f_{12}, \;\; B_{22}^e = 0, B_{23}^e = \frac{l_3}{12\Delta^e}f_{23}$$
$$B_{31}^e = 0, \qquad B_{32}^e = -\frac{l_2}{12\Delta^e}f_{13}, \; B_{33}^e = -\frac{l_3}{12\Delta^e}f_{23}$$
$$B_{41}^e = \frac{l_1}{6\Delta^e}(-f_{11} + f_{22})$$
$$B_{42}^e = \frac{l_2}{12\Delta^e}(-f_{11} - f_{12} + f_{13} + 2f_{23})$$
$$B_{43}^e = \frac{l_3}{12\Delta^e}(2f_{13} - f_{12} + f_{23} - f_{22})$$
$$B_{51}^e = \frac{l_1}{12\Delta^e}(f_{22} - f_{12} + f_{23} - 2f_{13})$$
$$B_{52}^e = \frac{l_2}{12\Delta^e}(-2f_{12} - f_{13} + f_{23} + f_{33})$$
$$B_{53}^e = \frac{l_3}{6\Delta^e}(f_{33} - f_{22})$$
$$B_{61}^e = \frac{l_1}{6\Delta^e}(2f_{23} - f_{13}), \quad B_{62}^e = \frac{l_2}{6\Delta^e}(f_{11} - f_{33}),$$
$$B_{63}^e = \frac{l_3}{12\Delta^e}(-f_{23} - 2f_{12} + f_{13} + f_{33})$$

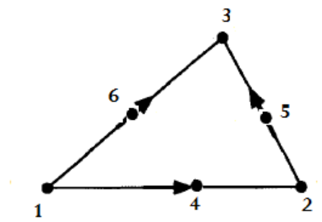

Fig. 1 Triangular element with linear edge and quadratic nodal element.

4. Numerical examples

Fig. 2 shows the cross section of the circular waveguide bounded by a perfect conductor filled with air with relative permittivity and permeability of approximately 1. The cross section of the circular waveguide is subdivided into 2048 triangular elements as shown in Fig. 3. Cut off wavenumber versus modes for linear edge element and quadratic nodal and lowest order nodal has been plotted in Fig 6. In order to illustrate the accuracy of this approach, the cut off wave number k_z of a cicular waveguide with unity radius are compared with analytical result [22] and hybrid first order, shown in Fig. 4. Notably, hybrid second order represented a higher accuracy than hybrid first order. The eigenvectors for some of the modes have been calculated for the electric field of corresponding modes Fig. 5.

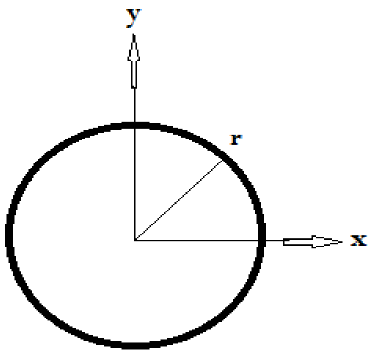

Fig. 2, Cross section of circular waveguide with radius r.

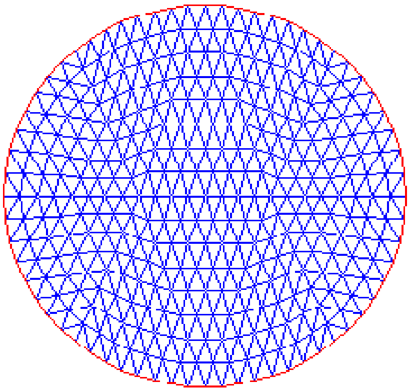

Fig. 3. Circular waveguide element division.

Table1. Comparison between cut-off wave number for circular waveguide.

Mode	Analytical[8]	MatlabCode	Comsol
TM10	2.405	2.406	2.411
TM11	3.832	3.839	3.841
TM12	5.136	5.154	5.159
TM13	6.380	6.418	6.433
TM20	5.520	5.543	5.560
TM21	7.016	7.065	7.083
TM22	8.417	8.503	8.522
TM23	9.761	9.892	9.941

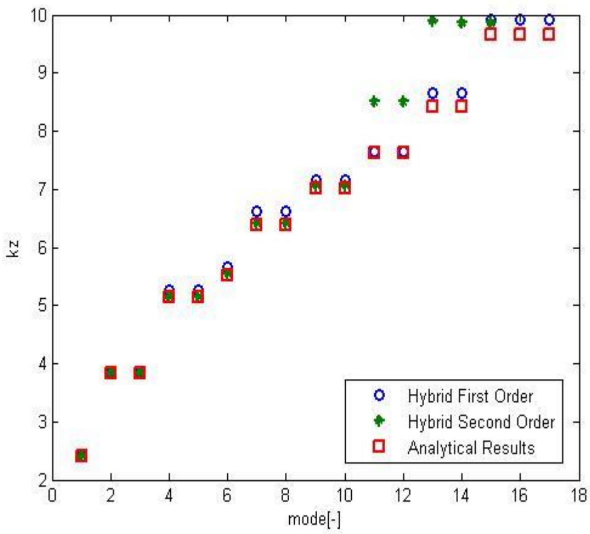

Fig. 4. Comparison between first order and second order nodal with edge element analysis and analytical results

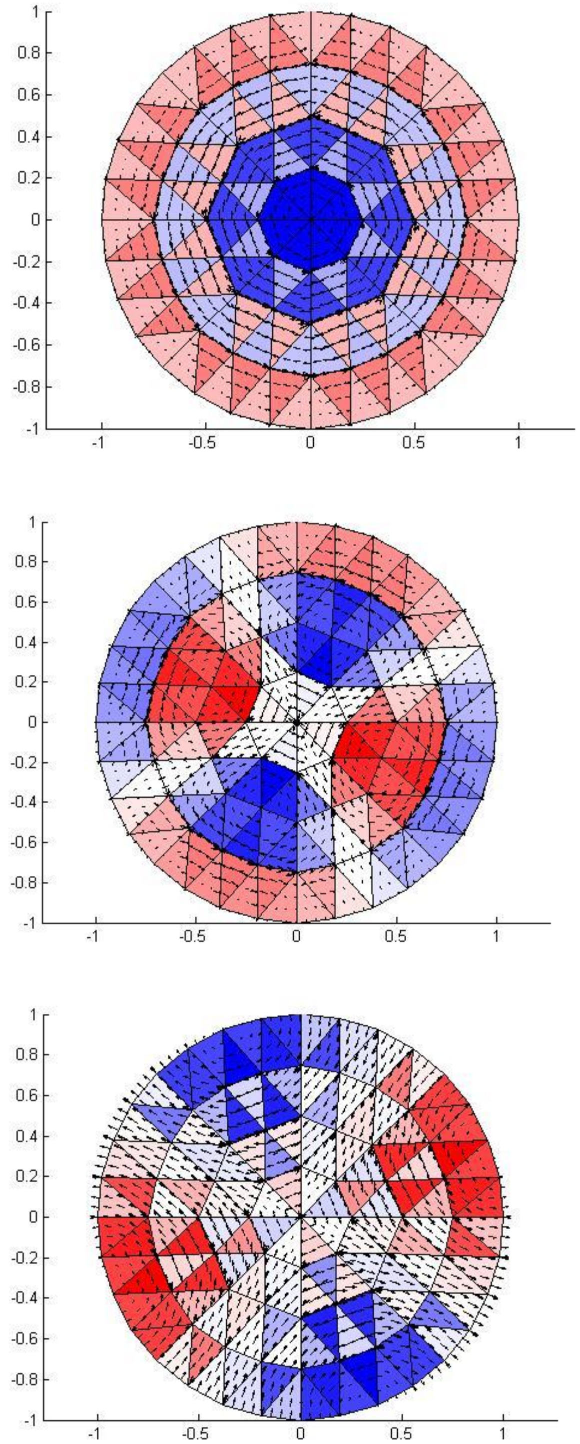

Fig. 5. Field configuration of some modes for circular waveguide.

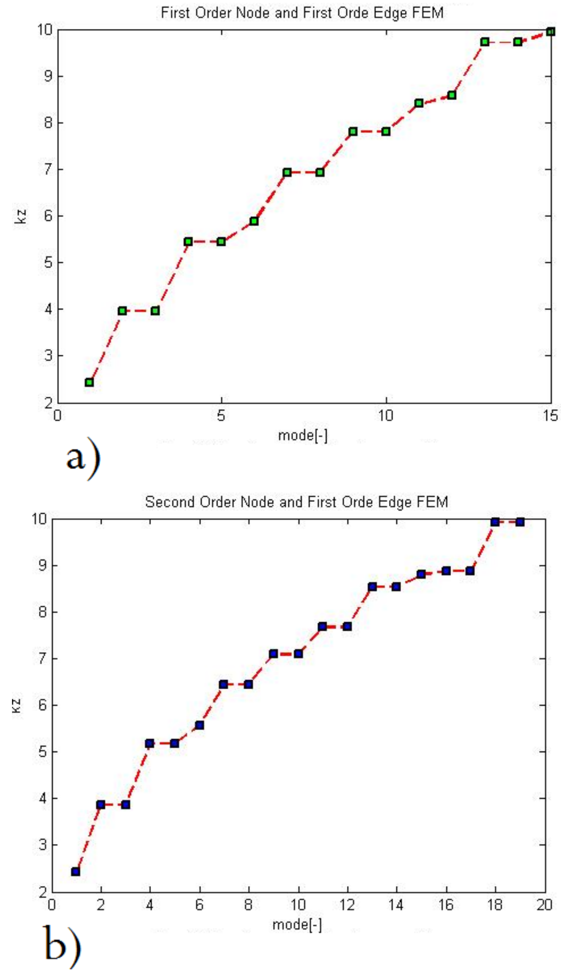

Fig. 6, Cut off wave numbers for circular waveguide, a) using first order nodal and linear edge element, b) using second order nodal and linear edge element.

5. Conclusion

A simple and effective finite element method for the analysis of microwave and optical waveguides was investigated using edge element method with the second-order mixed-interpolation-type triangular elements. It is a combination of linear edge elements for transverse components of the electric and quadratic nodal elements for the axial one. To eliminate spurious solutions and to be applicable for different shaped waveguides, triangular edge elements were utilized. An eigenvalue equation derived from the use of mixed-interpolation-type elements provides a direct solution for the propagation constant. The accuracy of this approach is investigated by comparing it with analytical results found in literature [8], Comsol software and first order hybrid. Notably, hybrid second order represented a higher accuracy than hybrid first order.

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