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Irrotational Motion of an Incompressible Fluid Past a Wing Section in an Unbounded Region

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Objectives

Address title problem so as to...

1. Formulate boundary conditions applicable in an unbounded region;
2. Specify conditions suitable to ensure a unique solution in a doubly connected region; and
3. Allow the interior boundary to have a sharp edge, such as a cusp;
4. Compare the COMSOL results with an exact solution.

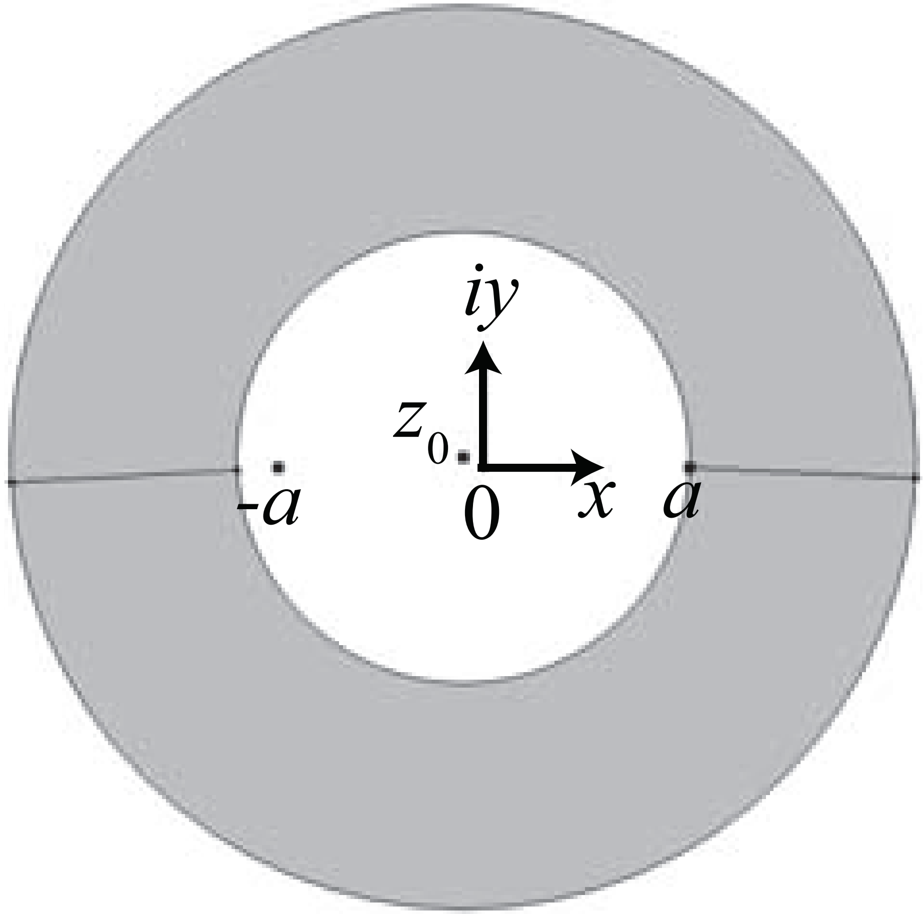

Figure 1. Bounded neighborhood of a disk containing the point $z = a > 0$ in the plane of the complex position coordinate, $z := x + iy$. The center, z_0, is at $(-0.1a, 0.05a)$

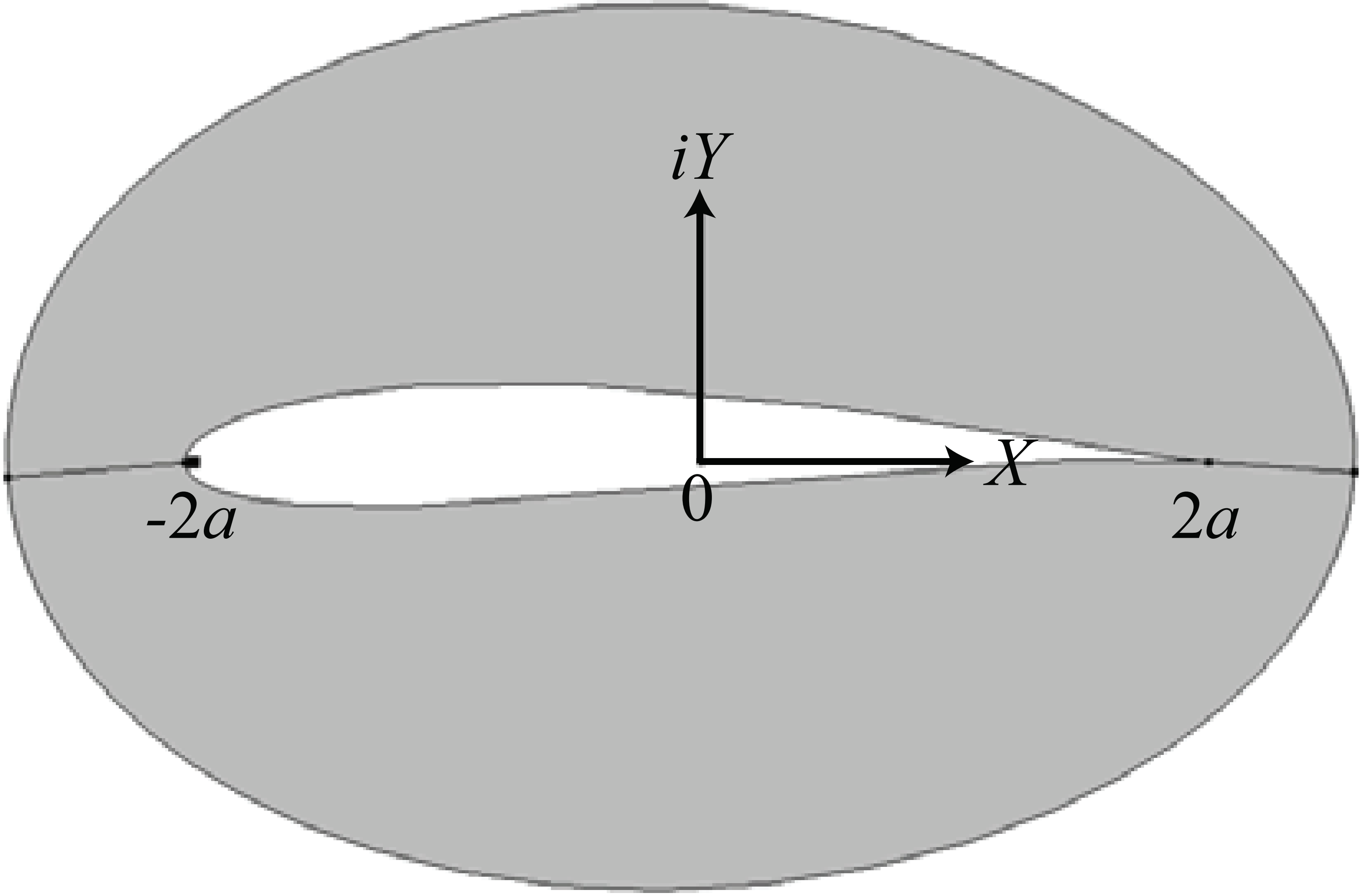

Figure 2. Bounded neighborhood of a airfoil containing the point $Z = 2a > 0$ in the plane of the complex position coordinate, $Z := X + iY$. Here $Z = z + a^2/z$, or $z^2 - 2Zz + a^2 = 0$.

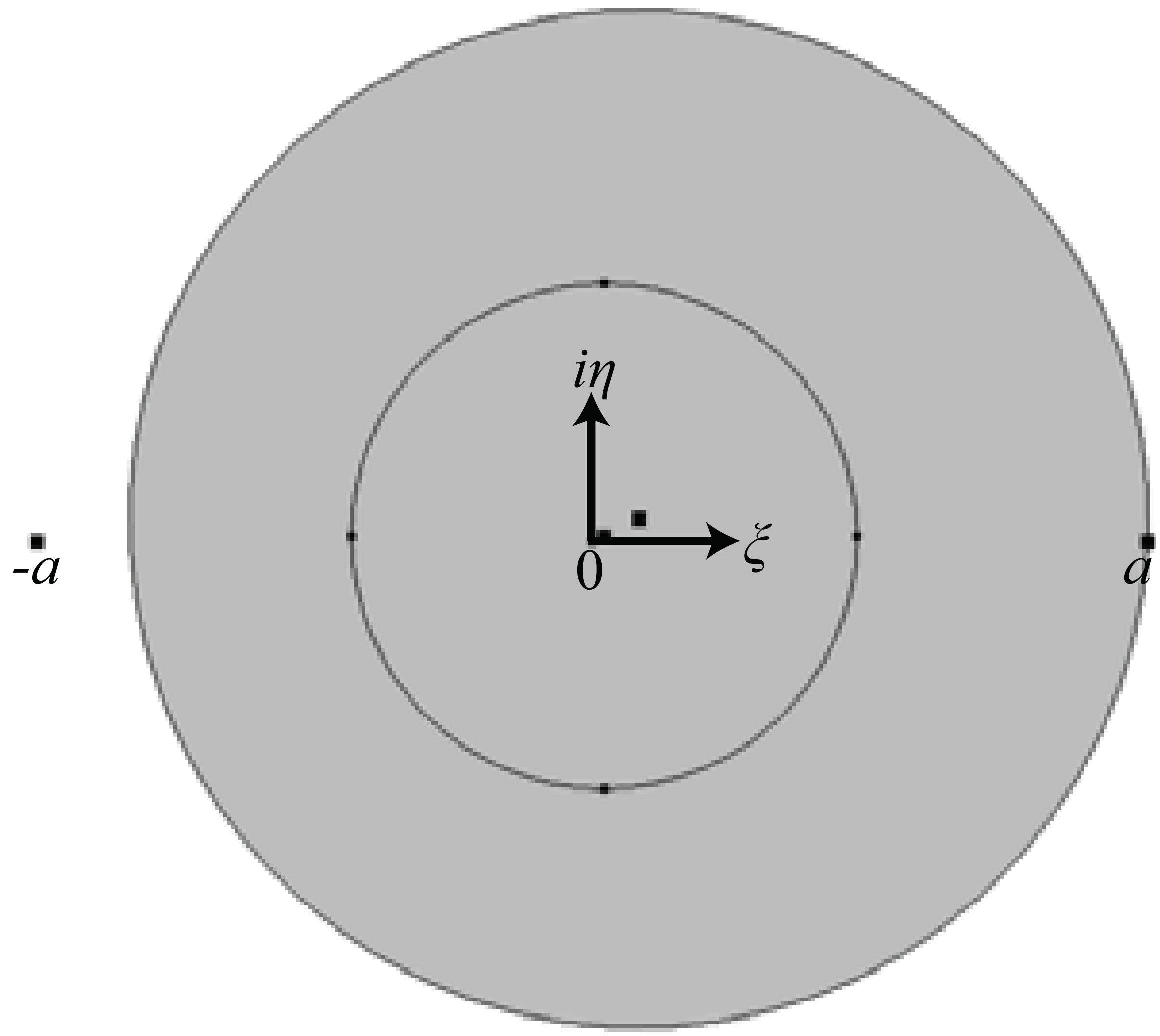

Figure 3. Image of the whole exterior of the airfoil in the plane of $\zeta = \xi + i\eta$. Here $Z = \zeta + a^2/\zeta$, or $\zeta^2 - 2Z\zeta + a^2 = 0$. The inner circle here corresponds to the outer edge in Fig. 2

Representation of the physics

Let (u, v) be the cartesian components of fluid velocity. I assume:

1. $u_x + v_y = 0$ (The fluid is incompressible);
2. $v_x - u_y = 0$ (The motion is irrotational).

Note that the foregoing are the real and imaginary parts of the complex equation

$$\frac{\partial}{\partial x}(u - iv) = \frac{d(u - iv)}{d(x + iy)} = \frac{1}{i}\frac{\partial}{\partial y}(u - iv)$$

(*i.e.* the Cauchy-Riemann equations)

The complex potential, $w = \phi + i\psi$

Let $(x, y) \mapsto \phi$ and $(x, y) \mapsto \psi$ be twice differentiable. Then the representations

$$(u, v) = (\phi_x, \phi_y) \qquad , \qquad (u, v) = (\psi_y, -\psi_x)$$

satisfy $v_x - u_y = 0$ and $u_x + v_y = 0$, respectively, and ϕ and ψ are called the velocity potential and stream function. Again, the foregoing are the real and imaginary parts of the complex equation

$$\frac{\partial}{\partial x}(\phi + i\psi) = \frac{d(\phi + i\psi)}{d(x + iy)} = \frac{1}{i}\frac{\partial}{\partial y}(\phi + i\psi)$$

$$= u - iv$$

Invariance conditions and impermeable-wall condition

At corresponding points in the ζ- and z-planes, I impose the condition:

$$u^{(\zeta)} - iv^{(\zeta)} = u^{(z)} - iv^{(z)},$$

while at corresponding points in the z- and Z-planes, I impose the condition:

$$w^{(Z)} = w^{(z)}.$$

Note that $\mathbf{u} \bullet \nabla\psi = 0$ by construction, so contours of constant $\psi = \Im(w)$ are parallel to $\mathbf{u}$ and therefore tangent to the airfoil surface.

The free-stream condition

I impose the condition

$$\lim_{\zeta \to 0} \{u^{(\zeta)} - iv^{(\zeta)}\} = Ue^{-i\alpha} \, ,$$

in which $U > 0$ and α denote the fluid speed in the remote free stream and the angle of attack, respectively.

THE KUTTA CONDITION (1902)

Owing to the double connectedness of the region occupied by fluid in the Z- and z-planes the problem posed thus far is insufficient to determine the solution uniquely. One fixes the solution only by fixing its circulation, $\Gamma := \oint_C \mathbf{u} \bullet d\mathbf{x}$, in which C is a loop that embraces the inner boundary once. One may fix Γ by imposing the condition that $u^{(Z)} - iv^{(Z)}$ be finite at the trailing edge, $Z = 2a$, or, equivalently, that $u^{(\zeta)} - iv^{(\zeta)}$ be zero at the $\zeta = a$.

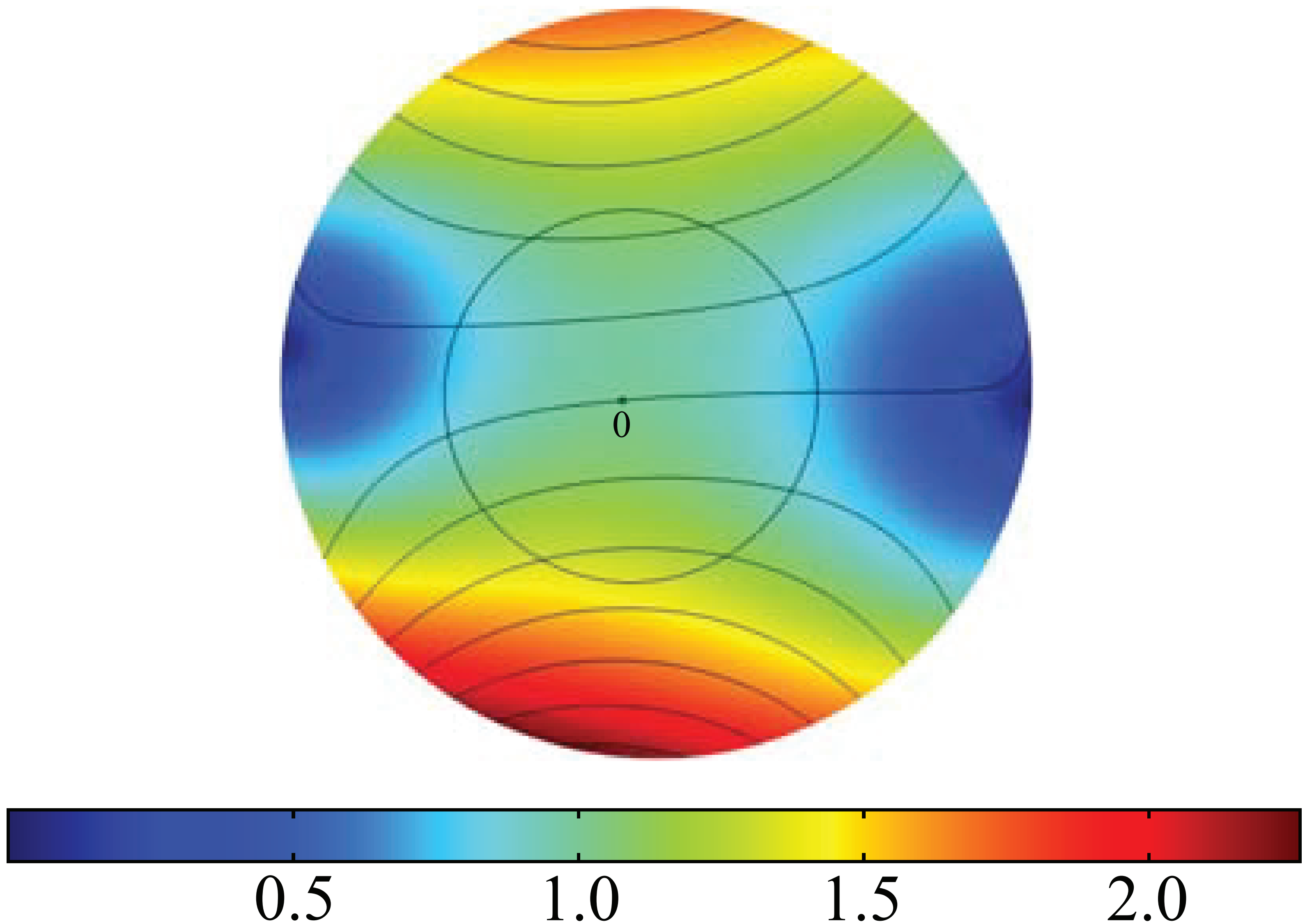

Figure 4. **Distribution of $|(u^{(\zeta)} - iv^{(\zeta)})/U|$ in the ζ-plane for $\alpha = 5°$**

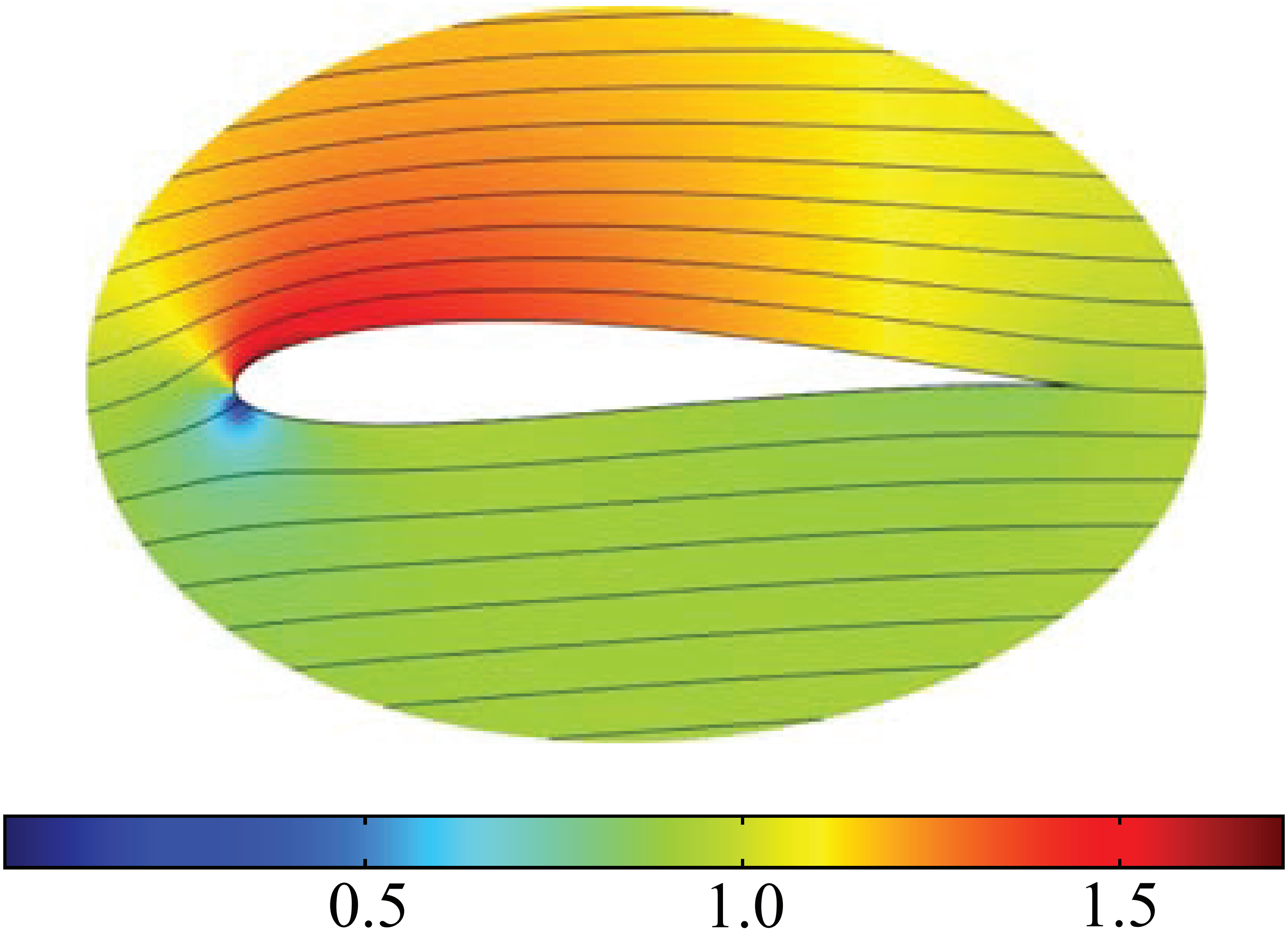

Figure 5. Distribution of $|(u^{(Z)} - iv^{(Z)})/U|$ in the Z-plane for $\alpha = 5^\circ$

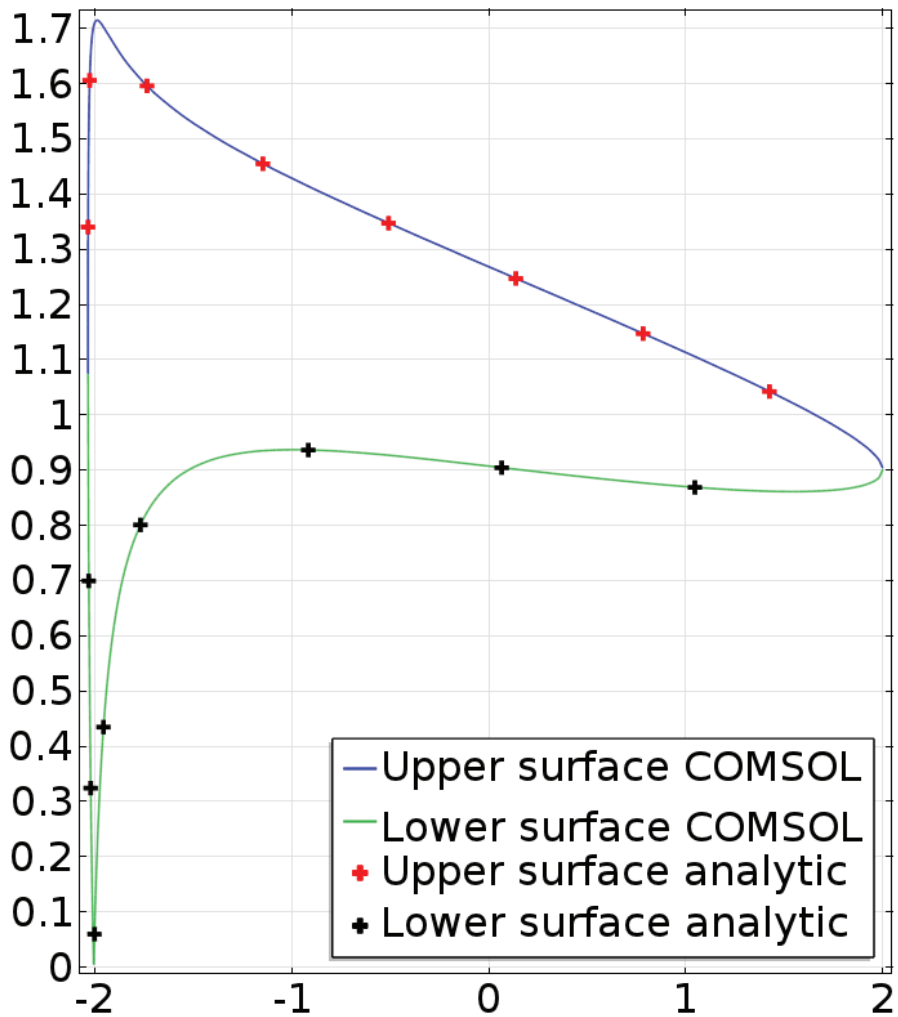

Figure 6. Distribution of $|(u^{(Z)} - iv^{(Z)})/U|$ over the airfoil surface versus X/a for $\alpha = 5^{\circ}$

Conclusions

1. COMSOL's Weak Form PDE physics interface enables convenient solution of the Cauchy-Riemann equations for the velocity field $\xi + i\eta \mapsto u^{(\zeta)} - iv^{(\zeta)}$ and the PDEs for the stream function for a given velocity field;
2. COMSOL's General Extrusion model coupling operators enable convenient replotting of data under changes of independent variables defined by conformal transformations.