Dynamics of Slender Structures

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Abstract:

The present work formulates the dynamics of a slender structure as a long beam. This results in a PDE that is fourth-order in space, and second order in time. In COMSOL the coupling method is used to decompose this problem into two second-order equations, one for displacements, and one for bending moments, which are then solved using the PDE, coefficient-mode approach; the eigenvalue solver is used to obtain frequencies and mode shapes.

For verification of approach, a simply supported uniform beam is considered, where the descriptive PDE's are rendered non-dimensional in terms of scaled variables, and for a beam length of 1. COMSOL computation then results in a fundamental lowest-order frequency that exactly matches the analytical result of π^2.

Keywords: Slender, structures, dynamics, vibration, PDE, beams.

1. Introduction

A variety of applications involve the dynamics or vibration of "slender structures" where the length of the structure is much greater than the lateral dimension. This may apply to tall buildings, or to process structures where slenderness ratios of 50 or more are not unusual. These structures may be subjected to various types of dynamic forcing that result in deformations and stresses. Therefore, it is important to determine characteristic, or natural frequencies of the structures. Because of the slenderness, these structures are amenable to modeling by one-dimensional beam-theory.

A slender circular vessel is considered where frequencies and mode shapes are determined with the eigenvalue solver. Changes in boundary conditions from a multiply-supported continuous beam have large effects upon the eigenvalues. Also, separate "free-floating" masses are coupled to the structures, and the behavior investigated. The use of beam theory in conjunction with the COMSOL PDE/eigenvalue solver is very effective. Solutions are very rapid, and the effects of parameters are easily determined.

2. Problem Formulation

In essence, very slender structures deform laterally as a beam, subject to various forces and constraints. Thus, beam theory may be employed to determine basic characteristics. For the beam mass in Fig. 1, let V = shear force, q = load per unit length, ρ = density, and A = cross-sectional area;

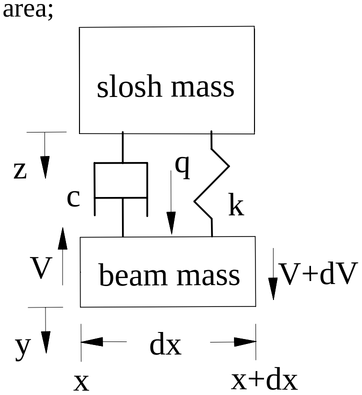

Figure 1: Problem Definition.

then application of $F = ma$ yields the force-balance

$$dV + qdx = (\rho A dx)\frac{d^2 y}{dt^2} \qquad \text{(1a)}$$

Dividing through by dx, this becomes the Euler beam equation, or the partial differential equation[1]:

$$\frac{\partial^2 M}{\partial x^2} \equiv \frac{\partial V}{\partial x} = -q + \rho A \frac{\partial^2 y}{\partial t^2} \qquad \text{(1b)}$$

where the moment, M, is related to the deflection curvature by[1]

$$M = -EI\frac{\partial^2 y}{\partial x^2} \qquad \text{(2)}$$

Thus, combining (1) and (2), we have the fourth-order differential equation for deflection, y, in terms of x and t:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = q - \rho A \frac{\partial^2 y}{\partial t^2} \qquad \text{(3a)}$$

In particular, if the load is $q = 0$, we obtain the eigenvalue equation

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = -\rho A \frac{\partial^2 y}{\partial t^2} \qquad \text{(3b)}$$

Here it is seen that y can be multiplied by any constant and still satisfy (3b); therefore, an eigenvalue solution gives only relative magnitudes, not absolutes.

In some situations, the beam may be connected to a sloshing material that has mass but no bending restraint, such as a dense fluid in a thin tube. In this case the interactive sloshing force in (3a) may be written as

$$q = k_s(z - y) + c_s(\dot{z} - \dot{y}) \qquad (3c)$$

Where k_s is an elastic constant with units of pressure, and c_s is a damping constant with units of viscosity; the equation of motion for the slosh mass is then

$$m_s \frac{\partial^2 z}{\partial t^2} = k_s(y - z) + c_s\left(\frac{\partial y}{\partial t} - \frac{\partial z}{\partial t}\right) \qquad (3d)$$

Where $m_s = \rho_s A_s$ is the slosh mass per length. The slosh natural frequency is then $\omega_s = \sqrt{k_s/m_s}$ and the critical damping is $c_{s,c} = 2\sqrt{k_s m_s}$; if this represents a fluid with viscous effects, only, with $k_s = 0$, then (3d) has no resonance frequency.

2.1 COMSOL Implementation

In COMSOL, the time-dependent one-dimensional equation can be solved:

$$e_a \frac{\partial^2 u}{\partial t^2} + d_a \frac{\partial u}{\partial t} - c_a \frac{\partial^2 u}{\partial x^2} + au = f_a \qquad (4a)$$

where u is a vector of variables, f_a is the forcing function, and e_a, d_a, c_a and a are coefficients. If this system has harmonic solutions, then $\partial^2 u / \partial t^2 = \lambda^2 u$ and $\partial u / \partial t = -\lambda u$, where λ are the eigenvalues to be determined; for $f_a = 0$, (4a) is then written as

$$-c_a \frac{\partial^2 u}{\partial x^2} + au = d_a \lambda u - e_a \lambda^2 u \qquad (4b)$$

Equations (4b), can be solved directly in COMSOL using the eigenvalue solver. For the present model, the equations in eigenvalue form are

$$\frac{\partial^2 M}{\partial x^2} = \rho A \lambda^2 y \quad (y\text{-equation}) \qquad (5a)$$

$$EI \frac{\partial^2 y}{\partial x^2} + M = 0 \quad (M\text{-equation}) \qquad (5b)$$

For the solution of these equations, the frequency in Hz is given by $f = -i\lambda/2\pi$ where $i = \sqrt{-1}$.

For testing this model in COMSOL on a beam of unity length it is convenient to scale $\xi = x/L$, $\eta = y/L$, $m = ML/EI$, and $\tau = t/t_c$, where t_c is the time constant to be determined; then for constant section properties, EI, we obtain

$$\frac{\partial^2 m}{\partial \xi^2} + \frac{\partial^2 \eta}{\partial \tau^2} = 0 \quad (y\text{-equation}) \qquad (6a)$$

$$\frac{\partial^2 \eta}{\partial \xi^2} + m = 0 \quad (M\text{-equation}) \qquad (6b)$$

where the time constant is

$$t_c = \sqrt{\frac{\rho A L^4}{EI}} = L^2 \sqrt{\frac{\rho A}{EI}} \qquad (6c)$$

The coupled PDE's, (6a) and (6b), can be solved directly in COMSOL using the eigenvalue solver.

3. Results

3.1 Scaled, Non-dimensional Test Beam

Consider a simply-supported beam of unit length; then, Eqns.(6) with the boundary conditions, $\eta = 0$ and $m = 0$, at both ends, yield the blue mode-shape in Fig. 1 with an eigenvalue of -9.8696i, or a non-dimensional frequency of 9.8696, which is precisely π^2 of the exact solution[2]. For a cantilever beam with boundary conditions, $\eta = 0$ at $\xi = 0$, and $m = 0$ at $\xi = 1$, the red mode-shape in Fig. 1 is obtained with a non-dimensional frequency of 3.515, which also corresponds to an exact solution[2].

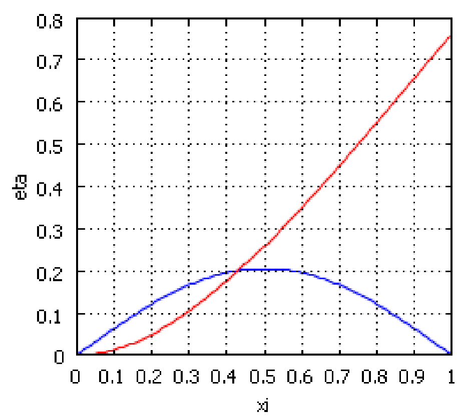

Figure 2: Mode Shapes for Scaled Beam.

3.2 Continuous Beam

A beam which has multiple supports along its span is referred to as a "continuous beam". Such cases are not easily analyzed analytically; however, with COMSOL and the beam approximation, solutions are quickly obtained. Continuous beam geometry occurs for the tubes of heat exchangers which may execute damaging vibrations due to vortex shedding[3]; here, the natural frequencies of the tube/support system seek to avoid the critical vortex shedding frequencies.

As an example, a 40 mm dia. by 1.9 m long tube was modeled as a half-tube, with profile shown Fig. 2(a):

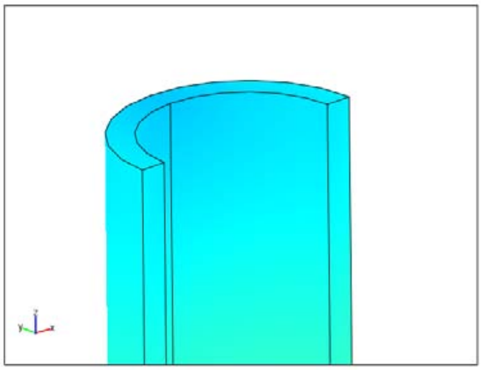

Figure 3(a): Heat Exchanger Tube Model.

The tube, clamped at one end and with 3 supports, was meshed and solved in 3-D using the COMSOL-structural and eigenvalue solver; the mode-shape is shown in Fig. 2(b):

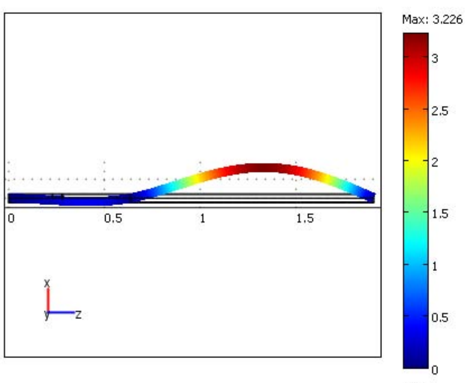

Figure 3(b): Fundamental Mode of 3-D Tube.

where the fundamental eigenvalue and frequency were -565.6i and 90.0 Hz. The corresponding beam solution is shown in Fig. 3(c):

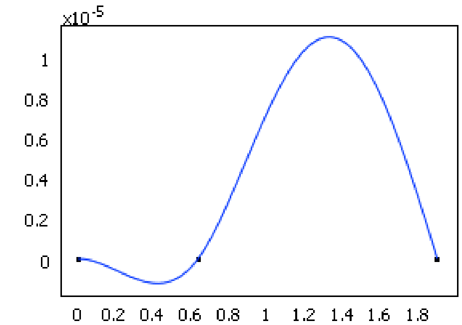

Figure 3(c): Fundamental Mode of Continuous Beam.

where the fundamental eigenvalue and frequency were -551.5i and 87.8 Hz.

Although the structural model was convenient to use in this model, for longer (more slender) objects, the beam model is more convenient and faster running. For example, Fig. 4 shows a 7 m long continuous beam with 7 supports; its fundamental eigenvalue and frequency were determined as -356.8i and 56.8 Hz (blue mode shape), and the first harmonic were -418.2i and 66.6 Hz (red mode shape):

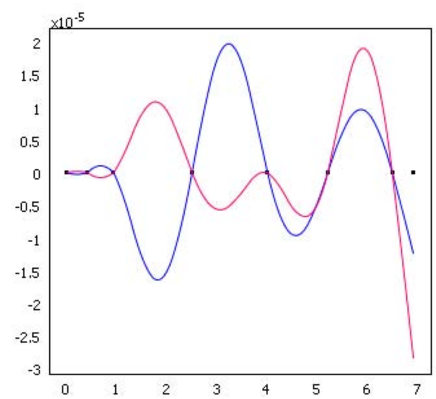

Figure 4: Continuous Beam Vibration Modes.

In Figs. 3 and 4, as in all modal analysis, it is the shapes and frequencies that are important. The actual displacement shown is arbitrary without a forcing function.

3.2 Effect of Sloshing

When a liquid or other substance dynamically interacts with the structure, then the combined natural frequency of the system is changed. For input to the PDF solver, it is convenient to write equations (1) to (3) in the matrix form:

$$-\begin{bmatrix} 0 & -1 & 0 \\ -EI & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} y'' \\ M'' \\ z'' \end{Bmatrix}+\begin{bmatrix} -k_s & 0 & k_s \\ 0 & 1 & 0 \\ k_s & 0 & -k_s \end{bmatrix}\begin{Bmatrix} y \\ M \\ z \end{Bmatrix}= \quad (7)$$

$$-\begin{bmatrix} -c_s & 0 & c_s \\ 0 & 0 & 0 \\ c_s & 0 & -c_s \end{bmatrix}\begin{Bmatrix} \dot{y} \\ \dot{M} \\ \dot{z} \end{Bmatrix}-\begin{bmatrix} -\rho A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -m_s \end{bmatrix}\begin{Bmatrix} \ddot{y} \\ \ddot{M} \\ \ddot{z} \end{Bmatrix}$$

where the prime and dot denote partial differentiation with respect to x and t, respectively; then the matrices correspond to the COMSOL coefficients *c*, *a*, d_a and e_a.

For example, if for the above model k_s = 0.1 psi (~700 Pa) and c_s = 0.001 Pa.s, then the fundamental eigenvalue is changed to 0.00015-14.4i, with a frequency of ~2.2 Hz, and the displacement of z (red curve) is orders greater than y (blue curve):

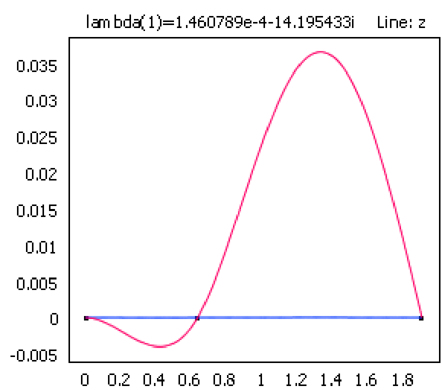

Figure 5: Sloshing Mode Shape.

4. Conclusions

The dynamics of slender structures was formulated in terms of conventional beam theory, which resulted in a fourth-order PDE; these equations were entered into the COMSOL PDE coefficient form as two coupled second-order equations.

For a scaled simply-supported beam, and a cantilever beam, it was shown that the COMSOL solution for fundamental natural frequency corresponded precisely to published analytical results. Furthermore, a continuous-beam model was found to be in excellent agreement with a two-dimensional structural model. For very slender structures with multiple supports, the beam model was expedient to use with the COMSOL PDE and eigenvalue solvers.

Modeling was extended to structures connected to a sloshing or free-moving mass. It was shown that such connection can lead to significant changes in the combined vibration frequency.

5. References

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6. Acknowledgements

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