Excerpt from the Proceedings of the 2011 COMSOL Conference in Stuttgart

LINKING THE DIMENSIONS

ANDREAS HELFRICH-SCHKARBANENKO*,[1], MARC MITSCHELE[2],
SEBASTIAN RITTERBUSCH[1], VINCENT HEUVELINE[1]

[1]Institute for Applied and Numerical Mathematics 4, [2]Institute for Analysis
*Corresponding author: helfrich@kit.edu, Karlsruhe Institute of Technology (KIT)
Engineering Mathematics and Computing Lab (EMCL) & Institute for Applied and Numerical Mathematics 4
Fritz-Erler-Str. 23, Geb. 01.86
76133 Karlsruhe, GERMANY

1. **Abstract.** In many applications it is sufficient to know the solution of a 3D boundary value problem in a cross-section. This paper introduces a concept how to model a 3D problem as a 2D problem. We apply this procedure to a boundary value problem which arises in the electrostatics. The presented method speeds up the direct problem solver, and can be used to improve the performance of the electrical impedance tomography. A numerical example completes this study.

2. **Introduction.** Let $\Omega_2 \subseteq \mathbb{R}^2$ and $\Omega_3 = \Omega_2 \times [-a, a]$ be bounded Lipschitz domains, see [1]. In many applications, e.g. in geoelectrics, or in medical and industrial fields, it is sufficient to know the solution of a 3D boundary value problem only in a cross-section $\Omega_2 \times \{0\}$, (*cross-sectional 3D problem*). The reduction of a cross-sectional 3D problem to a 2D problem would accelerate the computation but it takes more than only to reduce the dimension of the object to investigate. Additional input is essential. In [3] an approach was presented to face this challenge. In this paper we introduce another way to reduce the gap between the 2D and the 3D problems, and subsequently apply it on a problem, which arises in electrostatics as follows.

Problem 1. *Let $\gamma \in L^\infty(\Omega_3)$ be an electrical conductance*[1] *function with $0 < c < \gamma < C$. By $f : \Omega_3 \to \mathbb{R}$ we denote the current density and ν is the exterior unit normal on $\partial\Omega_3$. Find the electrical potential $u \in C^2(\Omega_3) \cap C(\overline{\Omega}_3)$ stimulated by f such that the steady-state diffusion equation and the Neumann boundary condition hold:*

$$\begin{aligned} -\mathrm{div}(\gamma\nabla u) &= f \quad \text{in } \Omega_3, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega_3. \end{aligned}$$

For the theoretical investigation of this problem and the numerical computation, we apply the Sobolev space, see [1],

$$\begin{aligned} H^1(\Omega_3) &:= W^{1,2}(\Omega_3) \\ &= \{u \in L^2(\Omega_3) : D^\alpha u \in L^2(\Omega_3)\ \forall\ |\alpha| \le 1\}, \end{aligned}$$

especially its subspace

$$H^1_\diamond(\Omega_3) := \{u \in H^1(\Omega_3) : \int_{\partial\Omega_3} u\, ds = 0\},$$

and consider the corresponding weak formulation of the problem 1

$$\int_{\Omega_3} \gamma \nabla u \cdot \nabla v\, dx = -\int_{\Omega_3} f v\, ds \quad \text{for all } v \in H^1_\diamond(\Omega_3)$$

obtained by means of the divergence theorem. For suitable $f \in H^{-1}(\Omega_3)$ it has an unique solution $u \in H^1_\diamond(\Omega_3)$, i.e. it is well-posed, see Lax-Milgram theorem in [1].

2.1. **Dimension Impact.** Now, for the simplified case $\gamma \equiv 1$, we will point out the effect of the space dimension on the solution in free space analytically. In the following, we denote by $\triangle_n$ the nth dimensional Laplacian, $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$ given by $\|x\| = (\sum_{k=1}^n x_k^2)^{1/2}$, δ_n is the n-dimensional Dirac's distribution and $E := \{x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3 : x_3 =$

Key words and phrases. electrostatics, 2D problem, 3D problem, accelerating algorithms.

[1] *Electrical admittance* is a complex-valued measure of how easily a circuit or device will allow a current to flow. Its unit is siemens (S). *Electrical conductance* is the real part of the admittance. It is the inverse quantity of the el. resistance and is measured in siemens, too.

0} a plane in $\mathbb{R}^3$. Applying Fourier transformation on the Poisson equation

$$-\triangle_n u_n(x) \;=\; \delta_n(x) \quad \text{for } x \in \mathbb{R}^n \tag{1}$$

yields the fundamental solution, see [2, Chap. 2.2.1]

$$\Phi_n(x) \;=\; \begin{cases} -\frac{1}{2\pi}\ln\|x\|, & x \in \mathbb{R}^2\backslash\{0\}, \\ \frac{1}{4\pi}\frac{1}{\|x\|}, & x \in \mathbb{R}^3\backslash\{0\}. \end{cases}$$

The fundamental solutions demonstrate clearly the dimension impact on the solution of the given BVP. As δ_n is not in $H^{-1}(\Omega_n)$, we consider more regular current densities f_n, and compute the electrostatic potential in $\mathbb{R}^3$ due to a ball as a support of f_n. By rescaling, it suffices to consider the case when the forcing function is

$$f_3(x) \;=\; \begin{cases} 1, & \text{for } \|x\| \le 1, \\ 0, & \text{else.} \end{cases}$$

We use the following well-known theorem.

Theorem 2.1. *A particular solution to the resulting Poisson equation* $-\triangle_3 \tilde{u}_3 \;=\; f_3$ *in* $\mathbb{R}^3$ *is given by a convolution integral*

$$\begin{aligned} \tilde{u}_3(x) \;&=\; \Phi_3 * f_3(x) \\ &=\; \frac{1}{4\pi}\int_{\|\xi\|_3<1} \frac{1}{\|x-\xi\|}\,d\xi. \end{aligned} \tag{2}$$

Since the forcing function f_3 is radially symmetric, the solution $\tilde{u} = \tilde{u}(r)$ is also radially symmetric. To evaluate the integral, we can choose $x = (0,0,x_3)^\top$, so that $r = \|x\| = |x_3|$. We use cylindrical coordinates $\xi = (\rho\cos\vartheta, \rho\sin\vartheta, \zeta)^\top$, so that

$$\|x-\xi\|_3 \;=\; \sqrt{\rho^2 + (x_3-\zeta)^2}.$$

The integral (2) can then be explicitly computed:

$$\begin{aligned} &\tilde{u}_3(x_3) \;= \\ &=\; \frac{1}{4\pi}\int_{-1}^{1}\int_0^{\sqrt{1-\zeta^2}}\int_0^{2\pi} \frac{\rho}{\sqrt{\rho^2+(x_3-\zeta)^2}}\,d\vartheta\,d\rho\,d\zeta \\ &=\; \frac{1}{2}\int_{-1}^{1}\sqrt{1+x_3^2-2x_3\zeta} - |x_3-\zeta|\,d\zeta \\ &=\; \begin{cases} \frac{1}{3|x_3|}, & |x_3| \ge 1, \\ \frac{1}{2} - \frac{x_3^2}{6}, & |x_3| \le 1. \end{cases} \end{aligned}$$

Therefore, by radial symmetry, the solution is

$$\tilde{u}_3(x) \;=\; \begin{cases} \frac{1}{3r}, & |r| \ge 1, \\ \frac{1}{2} - \frac{r^2}{6}, & |r| < 1. \end{cases} \tag{3}$$

Similarly, one computes the two-dimensional electrostatic potential and obtains the logarithmic asymptotic behavior of $\tilde{u}_2$.

3. **Linking the dimensions.** To solve a cross-sectional 3D problem, traditionally, we have following main options:

- apply the 3D model and extract the cross-section of interest,
- apply the Fourier transformation assuming the cylindrical property of the domain and γ,
- apply the boundary integral method for piecewise constant γ.

In this section, we propose a procedure to reduce the computational demands. The basic idea is to adapt a 2D model to the cross-sectional 3D model, in order to reduce the discrepancy between their solutions. In practice, our purpose is to expand the two-dimensional partial differential equation (PDE)

$$-\triangle_2 u_2(x) \;=\; f_2(x), \quad x \in \mathbb{R}^2,$$

by a function $h : \mathbb{R}^2 \to \mathbb{R}$, such that

$$-\triangle_2 \tilde{u}_3|_E(x) + h(x) \;=\; f_2(x), \quad x \in \mathbb{R}^2. \tag{4}$$

Here, h is the additional input we talk about in Section 2. Since

$$\begin{aligned} \frac{\partial}{\partial x_k}\frac{1}{\|x\|} \;&=\; -\frac{x_k}{\|x\|^3} \quad \text{and} \\ \frac{\partial^2}{\partial x_k^2}\frac{1}{\|x\|} \;&=\; -\frac{1}{\|x\|^3} + \frac{3x_k^2}{\|x\|^5}, \quad k = 1,2 \end{aligned}$$

and following the cases in (3), it yields for $\|x\| \ge 1$

$$\begin{aligned} -\triangle_2\tilde{u}_3(x) \;&=\; -\triangle_2\frac{1}{3\|x\|} \\ &=\; -\frac{1}{3}\left(-\frac{1}{\|x\|^3} + \frac{3x_1^2}{\|x\|^5} - \frac{1}{\|x\|^3} + \frac{3x_2^2}{\|x\|^5}\right) \\ &=\; \frac{1}{3}\left(\frac{2}{\|x\|^3} - \frac{3\|x\|^2}{\|x\|^5}\right) \\ &=\; -\frac{1}{3}\frac{1}{\|x\|^3}, \end{aligned}$$

and for $\|x\| < 1$

$$\begin{aligned} -\triangle_2\tilde{u}_3(x) \;&=\; -\triangle_2\left(\frac{1}{2} - \frac{\|x\|^2}{6}\right) \\ &=\; \triangle_2\frac{x_1^2+x_2^2}{6} \\ &=\; \frac{2}{3}. \end{aligned}$$

Hence, we finally obtain

$$h(x) \;=\; \begin{cases} \frac{1}{3}\frac{1}{\|x\|^3}, & \|x\| \geq 1, \\ -\frac{2}{3}, & \|x\| < 1. \end{cases}$$

So, in principal, by adding h we change the stimulation of the two-dimensional electrostatic potential in such a way that the resulting u_2^0 with

$$-\triangle_2 u_2^0(x) + h(x) \;=\; f_2(x), \quad x \in \mathbb{R}^2$$

and $\tilde{u}_3|_E$ coincide.

Remark 3.1. Note that $u_2^0 - \tilde{u}_3|_E \equiv 0$ holds only for $\gamma \equiv \mathit{const}$, i.e. for this case we found a perfect link between the 2D and the cross-sectional 3D Poisson PDE. For general γ, i.e. considering the operator $\mathrm{div}(\gamma\nabla\cdot)$, the gap between both solutions can not be closed completely, since the expansion of the two-dimensional PDE (1) by h was done for a homogeneous free space, i.e. we omit the effect on h of γ and the Neumann boundary condition, as well.

In the next section, we both solve the 2D BDE and the cross-sectional 3D BVP by means of Finite Element Method for constant and inhomogeneous conductance γ, and compare the results.

4. **Numerical Example.** We consider the domains

$$\Omega_2 \;:= [-1,\,1] \times [-1,\,1] \text{ and } \Omega_3 \;:= \Omega_2 \times [-1,\,1].$$

Let B_1, B_2 denote two balls with a radius 0.025 and the center points $p = (-1/2, -1/2, 0)^\top$ and $\bar{p} = (1/2, 1/2, 0)^\top$, respectively, see Fig. 1. Note that the center points are placed in Ω_2 which allows to take Ω_2 as a model for a cross-sectional 3D problem.

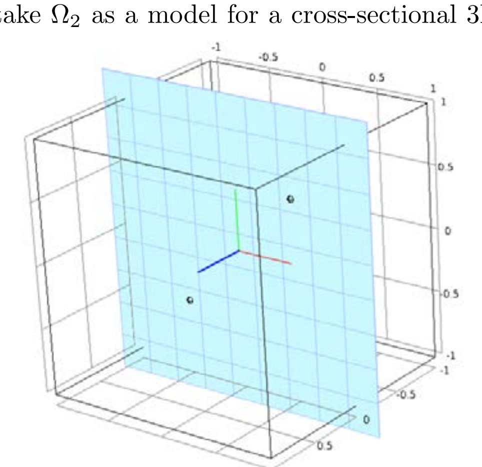

FIGURE 1. Geometry setup for the BVP

For the 3D problem, we set the following forcing function, i.e. the space charge density[2]:

$$f_3(x) \;=\; \begin{cases} 1, & \text{in } B_1, \\ -1, & \text{in } B_2, \\ 0, & \text{else.} \end{cases}$$

The 2D case of the forcing function $f : \Omega_2 \to \mathbb{R}$ is generated by $f_2(x_1, x_2) := f_3(x_1, x_2, 0)$.

4.1. **Homogeneous Conductance.** Firstly, we consider the simplest case, i.e. $\gamma \equiv 1$. Fig. 2 shows the two-dimensional electrostatic potential u_2.

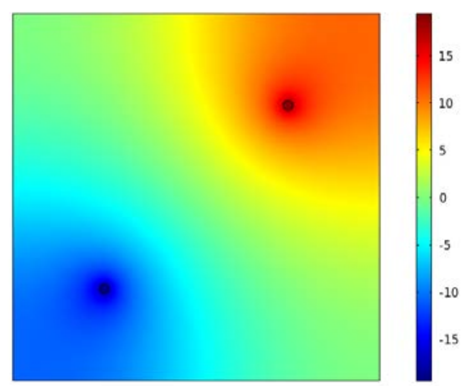

FIGURE 2. Solution u_2 of the 2D BVP

Its pendant, namely the potential $u_3|_{\Omega_3}$ is given in Fig. 3. The difference in behavior of the potentials u_2 and $u_3|_{\Omega_2}$ is clearly seen, note the color bar.

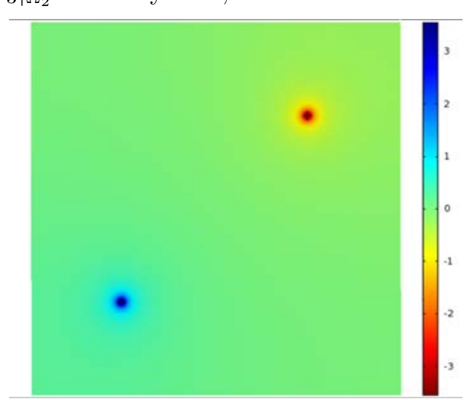

FIGURE 3. Solution $u_3|_{\Omega_2}$ of the 3D BVP restricted to the slice Ω_2

[2]COMSOL notation: es.rhod

In addition, applying the procedure introduced in Section 3 we solve numerically the 2D BVP for the adapted PDE

$$\operatorname{div}(\nabla u_2^h(x)) + h(x+p) - h(x+\bar{p}) \;=\; f_2(x),$$

see Fig. 4.

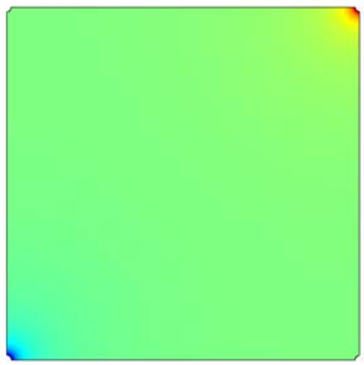

FIGURE 4. The solution u_2^h of a 2D BVP applying h.

4.2. **Inhomogeneous conductance.** Now we return to the origin problem (1) and assume small inhomogeneities in the electrical conductance function, i.e. we consider

$$\operatorname{div}(\gamma\nabla u_2^h(x)) + h(x+p) - h(x+\bar{p}) \;=\; f_2(x).$$

Note, that the forward operator, i.e. the map describing the relation between γ and the pair (u_n, f_n) is Lipschitz continuous w.r.t. γ, see [3]. That means, for fixed f small perturbations in γ involve small change in u_n. Thus, the procedure introduced is restricted not too strong, as seems in Section 2.1.

5. **Discussion.** In Remark 3.1, we emphasized that the function h does not satisfies the boundary condition. For special domain geometries and constant γ this defect can be corrected by means of Green representation theorem:

Theorem 5.1. *Let $u_n \in C^2(\Omega_n)$. Then, for $x \in \Omega_n$ it holds*

$$u_n(x) = -\int_{\Omega_n} \Phi(x-y)\triangle u(y)\,dx \;+$$
$$+\int_{\partial\Omega_n} \left(\Phi(x-z)\nabla_z u(z) - u(z)\nabla_z\Phi(x-z)\right)\cdot\nu\,ds(z).$$

That means, every solution of the Poisson equation $-\triangle u = f$ gets unique by the Neumann boundary condition $\partial u_n/\partial\nu$ and the Dirichlet boundary condition u_n on $\partial\Omega_n$. Applying Green's functions instead of the fundamental solutions, yields the Poisson formula for computing u_n for Neumann or Dirichlet BVP. For simple domains, like disc in $\mathbb{R}^2$ or a ball in $\mathbb{R}^3$, this representation is even analytical. So, for 2D Dirichlet BVP

Problem 2. *Let $K = \{x \in \mathbb{R}^2 : \|x\| < b\}$. Find $u_2 \in C^2(K) \cap C^1(\overline{K})$ such that*

$$\triangle u_2 = 0, \quad \text{in } K,$$
$$u_2 = g_2, \quad \text{on } \partial K.$$

the solution reads in polar coordinates

$$u_2(r,\theta) \;=\; \frac{a^2-r^2}{2\pi}\int_0^{2\pi} \frac{g_2(\theta')}{a^2 - 2ar\cos(\theta-\theta') + r^2}\,d\theta',$$

where $(r,\theta) \in [0,b] \times [0,2\pi)$. In spherical coordinates, the corresponding potential in a three-dimensional ball is given by

$$u_3(r,\theta,\varphi) = \frac{1}{4\pi} b^3\left(1 - \frac{\rho^2}{a^2}\right)$$
$$\cdot \int\!\!\int \frac{g_3(\theta',\varphi')\sin\varphi'}{(b^2+\rho^2-2b\rho\cos\theta)^{3/2}}\,d\theta'\,d\varphi',$$

with $(r,\theta,\varphi) \in [0,b] \times [0,\pi] \times [0,2\pi)$ and $\cos\theta = \cos\varphi - \cos\varphi' + \sin\varphi\sin\varphi'\cos(\theta-\theta')$. Using this formula, we can improve h and reduce further the difference between 2D and 3D problems.

6. **Conclusion and Future Study.** Considering the introduced idea in a setup of inverse problems, one can obtain a fast reconstruction algorithm. So, the new equation

$$\mathrm{div}(\gamma\nabla \mathrm{u}_2) + \mathrm{h} \;=\; \mathrm{f}_2$$

instead of

$$\mathrm{div}(\gamma\nabla \mathrm{u}_3) \;=\; \mathrm{f}_3$$

can be applied for solving an inverse cross-sectional 3D problem, i.e. to get $\gamma|_{\Omega_2}$ from boundary measurement $u_3|_{\partial\Omega_2}$ approximatively, however essential faster.

REFERENCES

[1] S. C. Brenner, L. R. Scott: *The Mathematical Theory of Finite Element Methods*, Springer, 1994.

[2] L. C. Evans: *Partial Differential Equations*, Graduate Studies in Mathematics, Volume 19, American Mathematical Society, 1998.

[3] A. Helfrich-Schkarbanenko: *Elektrische Impedanztomografie in der Geoelektrik*, Dissertation, Karlsruher Institut für Technologie, 2011.