Conducting Finite Element Convergence Studies Using COMSOL 4.0

Matthias K. Gobbert and David W. Trott

Department of Mathematics and Statistics
UMBC High Performance Computing Facility (HPCF)
Center for Interdisciplinary Research and Consulting (CIRC)
University of Maryland, Baltimore County
{gobbert,dtrott1}@umbc.edu
Acknowledgments: NSF, HPCF, CIRC, UMBC

See: Technical Report HPCF–2010–8, `www.umbc.edu/hpcf` > Publications

Problem Statement

- Problem: Assess the quality of a FEM solution quantitatively for all Lagrange elements with polynomial degrees $1 \leq p \leq 5$ available in COMSOL.
- Approach: Use guidance from the a priori error estimate

$$\|u - u_h\|_{L^2(\Omega)} \leq C\, h^q, \qquad \text{as } h \to 0$$

  with a constant C independent of h and the convergence order $q > 0$. Here, h is the maximum side length of the elements in the triangulation.
- Goal: Confirm that solutions on a sequence of meshes, that are progressively uniformly refined, behaves as predicted by the error estimate.
- Concrete goal: Show how to do this in COMSOL's GUI!

Computational Convergence Study

- Consider the FEM solution u_h on a sequence of meshes with uniform refinement levels $r = 0, 1, 2, \ldots$, and let $E_r := \|u - u_h\|_{L^2(\Omega)}$ denote the norm of the error.
- Then assuming that $E_r = C\,h^q$, the error for the next coarser mesh with mesh spacing $2h$ is $E_{r-1} = C\,(2h)^q = 2^q\,C\,h^q$. Their ratio is then $R_r = E_{r-1}/E_r = 2^q$ and $Q_r = \log_2(R_r)$ provides us with a computable estimate for q as $h \to 0$. Example:

r	E_r	R_r	Q_r
0	1.077e–01	N/A	N/A
1	2.652e–02	4.06	2.02
2	6.709e–03	3.95	1.98
3	1.684e–03	3.98	1.99
4	4.214e–04	3.99	2.00

- This indicates that the convergence order is $q = 2$.

FEM Theory for Lagrange Elements

- For linear Lagrange elements (polynomial degree $p = 1$), optimal convergence order is $q = p + 1 = 2$ in

$$\|u - u_h\|_{L^2(\Omega)} \leq C\,h^q = C\,h^2$$

- For Lagrange FEM with polynomial degree $p = 1, \ldots, 5$, as available in COMSOL, we expect $q = p + 1$ in

$$\|u - u_h\|_{L^2(\Omega)} \leq C\,h^q = C\,h^{p+1},$$

  provided that
  - the solution u is smooth enough: $u \in H^k(\Omega)$ with $k \geq p + 1$,
  - the domain Ω is open, bounded, convex, and simply connected,
  - and the domain boundary $\partial\Omega$ piecewise polygonal, i.e., the domain Ω can be triangulated without error.
- For Lagrange FEM with polynomial degree $p = 1, \ldots, 5$, if the solution is $u \in H^k(\Omega)$, then

$$\|u - u_h\|_{L^2(\Omega)} \leq C\,h^q, \quad q = \min\{k, p + 1\}.$$

Elliptic Test Problem: Problem Statement

Classical elliptic test problem on a polygonal domain with Dirichlet boundary conditions on $\Omega \subset \mathbb{R}^2$

$$\begin{aligned} -\triangle u &= f \quad \text{in } \Omega, \\ u &= r \quad \text{on } \partial\Omega. \end{aligned}$$

- Use unit square as domain: $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$.
- Right-hand side function:

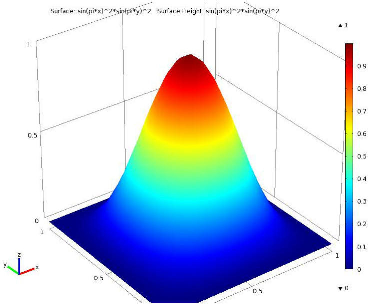
$$f(x,y) = (-2\pi^2)\ \big(\cos(2\pi x)\sin^2(\pi y) + \sin^2(\pi x)\cos(2\pi y)\big)$$

- Homogeneous Dirichlet boundary conditions:

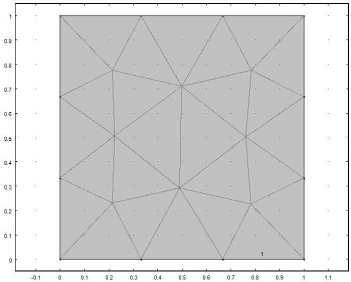
$$r(x,y) = 0$$

Elliptic Test Problem: PDE Solution

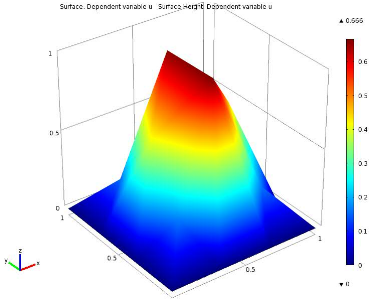
- $u(x,y) = \sin^2(\pi x)\,\sin^2(\pi y)$ on $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$
- u infinitely often differentiable $\Longrightarrow u \in H^k(\Omega)$ with $k = \infty$
- Therefore convergence order $q = \min\{k, p+1\} = p+1$.



Elliptic Test Problem: Mesh and FEM Solution with Order $p = 1$



Extremely Coarse Mesh



Linear Lagrange Elements ($p = 1$)

- This mesh has $N_e = 26$ elements and $N_v = 20$ vertices.
- DOF is equal to N_v for linear Lagrange elements.

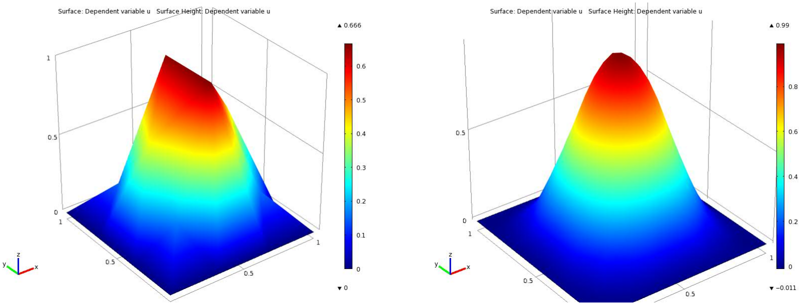
Elliptic Test Problem: Convergence Study with Linear Lagrange

Lagrange elements with $p = 1$

r	N_e	N_v	DOF	E_r^2	E_r	R_r	Q_r
0	26	20	20	1.160e–02	1.077e–01	N/A	N/A
1	104	65	65	7.031e–04	2.652e–02	4.06	2.02
2	416	233	233	4.501e–05	6.709e–03	3.95	1.98
3	1664	881	881	2.835e–06	1.684e–03	3.98	1.99
4	6656	3425	3425	1.776e–07	4.214e–04	3.99	2.00

- Same results as presented before. Additional information includes E_r^2 which is the raw data that appears in the GUI along with statistical information about the mesh.
- Note: Number of vertices N_v was obtained using LiveLink with MATLAB. See tech. rep. HPCF-2010-8.

Elliptic Test Problem: FEM Solutions with $p = 1$ and $p = 2$



Linear Lagrange ($p = 1$) Quadratic Lagrange ($p = 2$)

Elliptic Test Problem: Lagrange Elements of Orders $p = 2$ and $p = 3$

Lagrange elements with $p = 2$

r	N_e	N_v	DOF	E_r^2	E_r	R_r	Q_r
0	26	20	65	4.351e–05	6.596e–03	N/A	N/A
1	104	65	233	1.259e–06	1.122e–03	5.88	2.56
2	416	233	881	2.076e–08	1.441e–04	7.79	2.96
3	1664	881	3425	3.294e–10	1.815e–05	7.94	2.99
4	6656	3425	13505	5.180e–12	2.276e–06	7.97	3.00

Lagrange elements with $p = 3$

r	N_e	N_v	DOF	E_r^2	E_r	R_r	Q_r
0	26	20	136	6.991e–06	2.644e–03	N/A	N/A
1	104	65	505	2.031e–08	1.425e–04	18.56	4.21
2	416	233	1945	7.460e–11	8.637e–06	16.50	4.04
3	1664	881	7633	2.834e–13	5.327e–07	16.22	4.02
4	6656	3425	30241	1.095e–15	3.309e–08	16.10	4.01

Elliptic Test Problem: Lagrange Elements of Orders $p = 4$ and $p = 5$

Lagrange elements with $p = 4$

r	N_e	N_v	DOF	E_r^2	E_r	R_r	Q_r
0	26	20	233	6.634e–09	8.145e–05	N/A	N/A
1	104	65	881	1.467e–11	3.830e–06	21.27	4.41
2	416	233	3425	1.578e–14	1.256e–07	30.49	4.93
3	1664	881	13505	1.605e–17	4.006e–09	31.36	4.97
4	6656	3425	53633	1.595e–20	1.263e–10	31.71	4.99

Lagrange elements with $p = 5$

r	N_e	N_v	DOF	E_r^2	E_r	R_r	Q_r
0	26	20	356	7.656e–10	2.767e–05	N/A	N/A
1	104	65	1361	1.421e–13	3.770e–07	73.39	6.20
2	416	233	5321	3.421e–17	5.849e–09	64.45	6.01
3	1664	881	21041	8.306e–21	9.114e–11	64.17	6.00
4	6656	3425	83681	1.819e–24	1.349e–12	67.58	6.08

Conclusions and Live Demonstration

Conclusions:

- COMSOL: behaves as predicted by theory for Lagrange elements on triangular meshes in 2-D.
- Education: COMSOL can be used to demonstrate FEM theory
- Applications: tests of this type can guide choice of finite elements
- Limitation of GUI: convergence study entirely in the GUI of COMSOL; however, the refinement level r and polynomial degree p cannot be programmed as parameters in a parameter sweep $\Rightarrow$ consider using COMSOL's LiveLink for MATLAB!
- Support: tech. rep. HPCF–2010–8 at `www.umbc.edu/hpcf` > Publications, includes the mph-file and m-files for LiveLink for MATLAB

Demonstration:

- Loads mph-file as starting point: (i) sets up domain, PDE, BC; (ii) chooses linear Lagrange ($p = 1$) with 'extremely coarse' mesh and no refinement ($r = 0$); (iii) after solution gives 3-D view of solution and square of FEM error by post-processing integration
- Shows how to obtain refined meshes for $r = 1, 2, \ldots$ and their solutions including square of error