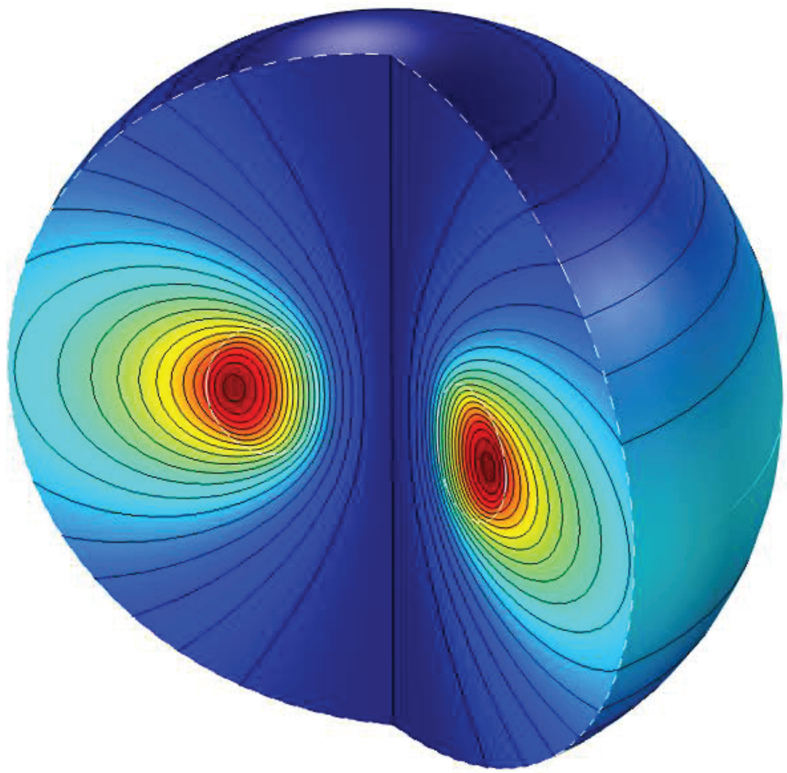
Structure Of A Classical Vortex Ring

John M. Russell, Professor Emeritus
Florida Inst. of Technology, Melbourne, Florida
COMSOL Conference, Boston, October 4, 2018

Objective

Develop a COMSOL simulation that incorporates complications such as those shown here:



SPHERICAL COORDINATES IN PHYSICAL AND PROXY DOMAINS VIA KELVIN INVERSION

	Physical domain	Conversion	Proxy domain
Radial	R	$Rq = a^2$	q
Colatitudinal	θ	$\theta = \vartheta$	ϑ
Azimuthal	ϕ	$\phi = \varphi$	φ

$$R\frac{\partial}{\partial R} = -q\frac{\partial}{\partial q} \quad , \quad \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \vartheta} \quad , \quad \frac{\partial}{\partial \phi} = \frac{\partial}{\partial \varphi}$$

$$\hat{\mathbf{e}}_R = \hat{\mathbf{u}}_q \quad , \quad \hat{\mathbf{e}}_\theta = \hat{\mathbf{u}}_\vartheta \quad , \quad \hat{\mathbf{e}}_\phi = \hat{\mathbf{u}}_\varphi$$

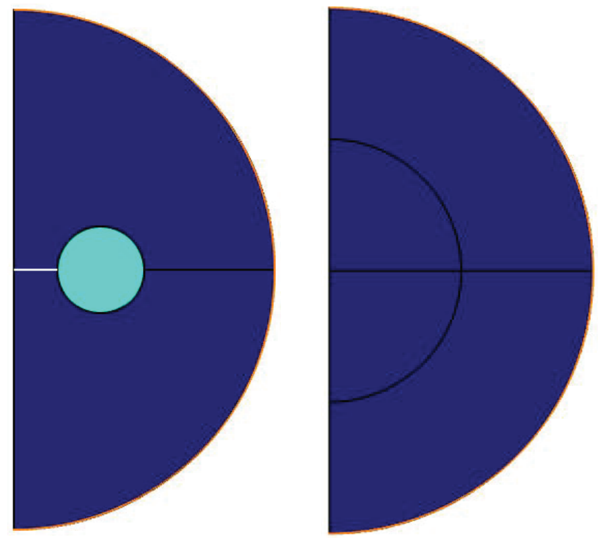


Figure 1 Schematic views of meridional sections of $\mathcal{R}^i$ (left panel) and $\mathcal{Q}$ (right panel). In the left panel the light blue region is a section of $\mathcal{R}^c$ and the dark blue region is a section of $\mathcal{R}^i \backslash \mathcal{R}^c$. The short white segment there is a section of the diaphragm, $\mathcal{S}^d$.

Cylindrical coordinates in physical and proxy domains via Kelvin Inversion

	Physical domain	Proxy domain
Tranverse	$r = R\sin\theta$	$\varpi = q\sin\vartheta$
Axial	$z = R\cos\theta$	$\zeta = q\cos\vartheta$

$$\hat{\mathbf{e}}_r = \hat{\mathbf{u}}_\varpi \quad , \quad \hat{\mathbf{k}} = \hat{\mathbf{u}}_\zeta$$

Meridional gradient operators:

$$\nabla^{(m)} := \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{k}} \frac{\partial}{\partial x} \quad , \quad \nabla_q^{(m)} := \hat{\mathbf{u}}_\varpi \frac{\partial}{\partial \varpi} + \hat{\mathbf{u}}_\zeta \frac{\partial}{\partial \zeta}$$

Field equations in three domains

In the vortex core, $\mathcal{R}^c$:

$$\nabla^{(\mathrm{m})} \bullet [(1/r)\nabla^{(\mathrm{m})}\Psi] = Ar \ ,$$

in which the velocity components (u_r, u_z) satisfy $u_r = (1/r)\partial\Psi/\partial z$, $u_z = -(1/r)\partial\Psi/\partial r$ and A is a constant. In the bounded region of irrotational motion, $\mathcal{R}^i \backslash \mathcal{R}^c$:

$$\nabla^{(\mathrm{m})} \bullet (r\nabla^{(\mathrm{m})}\Phi) = 0 \ ,$$

in which $u_r = \partial\Phi/\partial r$, $u_z = \partial\Phi/\partial r$. In the proxy exterior, $\mathcal{Q}$:

$$\nabla_q^{(\mathrm{m})} \bullet [\varpi(a^2/q^2)\nabla_q^{(\mathrm{m})}\Phi] = 0 \ .$$

Conditions on the portal

Let $\partial(\mathcal{R}^i \backslash \mathcal{R}^c)_>$ be the outer boundary of $\mathcal{R}^i \backslash \mathcal{R}^c$ and think of $\partial(\mathcal{R}^i \backslash \mathcal{R}^c)_>$ and $\partial\mathcal{Q}$ as two sides of a *portal*. Condition 1 there is a Dirichlet condition for Φ on $\partial\mathcal{Q}$, which equates it to the value of Φ on $\partial(\mathcal{R}^i \backslash \mathcal{R}^c)_>$. Condition 2 there is a Flux/Source boundary condition for Φ on $\partial(\mathcal{R}^i \backslash \mathcal{R}^c)_>$, namely

$$-\hat{\mathbf{n}} \bullet (r \nabla^{(\mathrm{m})} \Phi) = \frac{\varpi}{q} \left(\varpi \frac{\partial \Phi}{\partial \varpi} + \zeta \frac{\partial \Phi}{\partial \zeta} \right),$$

whose right member is the value of the left member after Kelvin Inversion and evaluation on $\partial\mathcal{Q}$.

Conditions on the boundary of the core

The vortex ring propagates with velocity $-W\hat{\mathbf{k}}$ ($W \geq 0$) and the boundary of its core in impermeable. On the *exterior* side of that boundary the impermeability condition amounts to a Flux/Source condition for Φ; On the *interior* side of that boundary the impermeability condition amounts to a Diriclet condition for Ψ.

Circulation, C, about the core and its relation to A

Consider an oriented close contour $\mathfrak{L}$ that embraces the core once and let u_t be the tangential component of the fluid velocity on $\mathfrak{L}$. Define C—called *the circulation about* $\mathfrak{L}$—by the integral $\int_{\mathfrak{L}} u_t \, ds$, in which ds is the differential arc length on $\mathfrak{L}$ and let δ be the radius of a circular disk whose area equals that of a typical cross section of the vortex core. One may then show that

$$A = C/(a\pi\delta^2) \ .$$

ON THE FAR FIELD DIPOLE STRENGTH, G

Let the fluid velocity, $\nabla\Phi$, satisfy $\nabla\Phi \longrightarrow \mathbf{0}$ as $R \longrightarrow \infty$. If, as here, there are no sources the far field behavior of Φ has the asymptotic form

$$\Phi = \Phi_\infty + \mathbf{G} \bullet \nabla[1/(4\pi R)] + O(R^{-3}) \ ,$$

in which $\mathbf{G} = \mathbf{G}_1 + \mathbf{G}_2$, in which

$$\mathbf{G}_1 := \iint\limits_{\mathcal{S}} \Phi \hat{\mathbf{n}} \, dA \quad , \quad \mathbf{G}_2 := -\iint\limits_{\mathcal{S}} \mathbf{R}(\mathbf{u} \bullet \hat{\mathbf{n}}) dA \ .$$

Here $\mathbf{G} = -G\hat{\mathbf{k}}$, $G > 0$.

Normalizations

Let a be the value of r at the centroid of a typical cross section of the ring. In the simulations reported herein I took $a = 1\,\text{m}$, $G = 1\,\text{m}^4/\text{s}$, and $\delta = a/2$.

Let $\mathbf{v}$ denote the fluid velocity as seen by an observer moving with the ring and let Δv_t be the corresponding slip velocity across the core boundary. In the problem posed herein Φ, W, and C are all bilinear functions of the plunge velocity, W and the circulation C.

$(C \quad W)^T$ AS THE SOLUTION OF A MATRIX EQUATION

From the foregoing bilinearity properties we have

$$G_C C + G_W W = G \ ,$$
$$(\Delta v_t)_C C + (\Delta v_t)_W W = \Delta v_t \ .$$

If the slip velocity at the inner equator, $(\Delta v_t)_{\mathrm{ie}}$, is set equal to zero and $G = 1\,\mathrm{m}^4/\mathrm{s}$ we have

$$\begin{bmatrix} G_C & G_W \\ (\Delta v_t)_{C\,\mathrm{ie}} & (\Delta v_t)_{W\,\mathrm{ie}} \end{bmatrix} \begin{pmatrix} C \\ W \end{pmatrix} = \begin{pmatrix} 1\,\mathrm{m}^4/\mathrm{s} \\ 0 \end{pmatrix} \ .$$

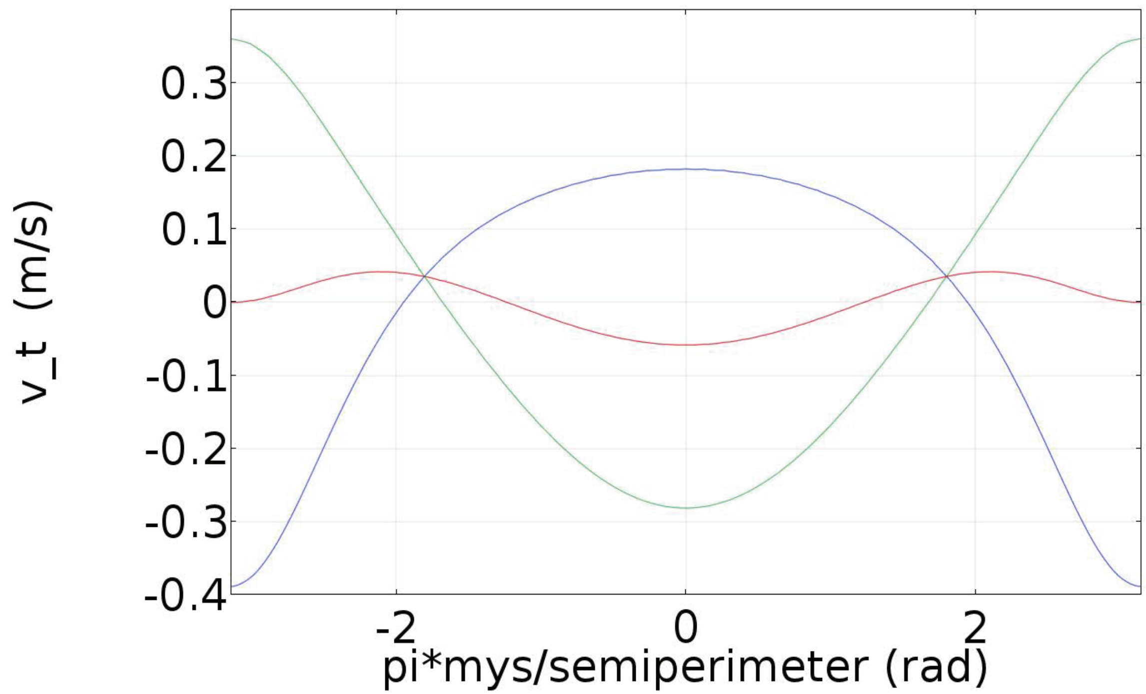


Figure 2 Distributions of the slip velocity across the core boundary in three cases, namely: circulation without plunge (green curve), plunge without circulation (blue cuve), and circulation and plunge with zero slip at inner equator (red curve)

Use of the Moving Mesh Interface

In the present simulaton COMSOL's Geometry Sequence generates a reference core whose cross section is a circular disk of area $\pi\delta^2$ and centroidal radius a. COMSOL's Moving Mech interface then transforms the reference core to one with a general noncircular cross section using formulas derived to ensure that the cross sectional area and the centroidal radius are again equal to $\pi\delta^2$ and a, respectively.

Control variables

This general noncircular cross section is a (translated and rescaled version of) the interior of the following parametric curve

$$R_{\partial 1} = a + P_\partial \cos\alpha \quad , \quad Z_{\partial 1} = P_\partial \sin\alpha \ ,$$

in which $P_\partial = \delta\left[1 + \sum_1^N \epsilon_n \cos(n\alpha)\right]$ and in which the coefficients ϵ_n, $n \in \{1, \ldots, N\}$ are shape parameters that will be control variables in an optimization problem.

Objective Function

The objective function F in the optimization problem is

$$F = C^{-2} \int_{\partial \mathcal{D}^c} (\Delta v_t)^2 ds \cdot \int_{\partial \mathcal{D}^c} ds$$

in which $\mathcal{D}^c$ is a typical meridional cross section of $\mathcal{R}^c$.

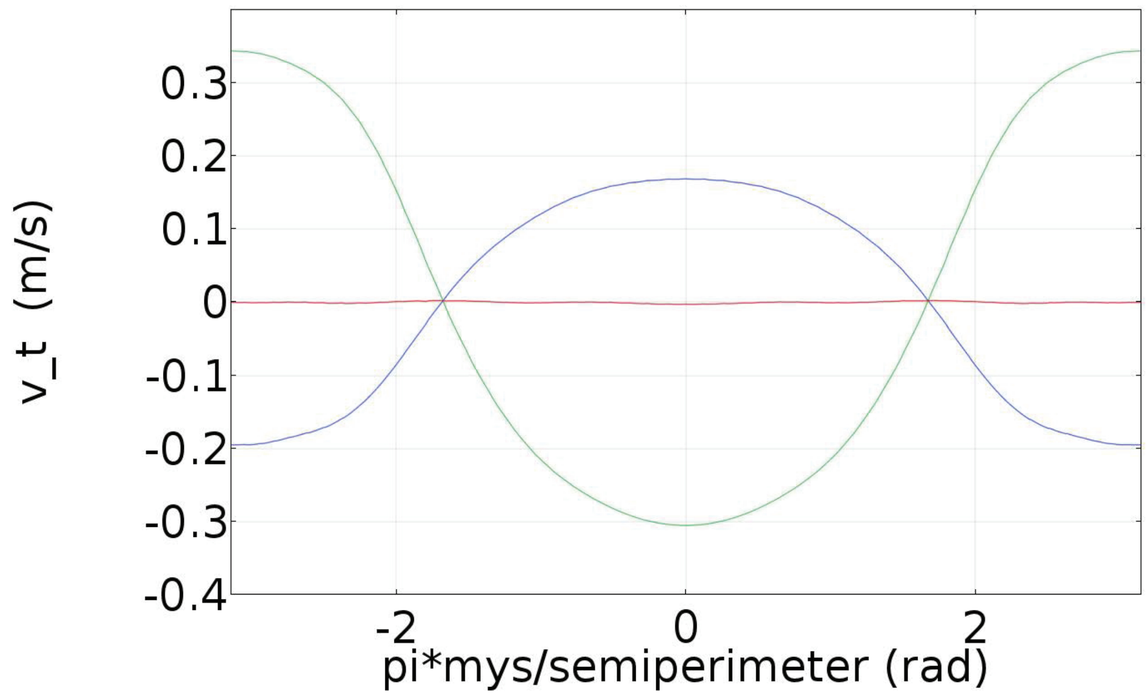


Figure 3 Legend same as that of Fig. 2 except that this time, the results shown are for a non-circular core shape after optimization to minimize slip across the core boundary in the case of circulation and plunge with zero slip at inner equator.

Computation of Ψ in $\mathcal{R}^i \backslash \mathcal{R}^c$

Having $u_r = \partial\Phi/\partial r$ and $u_z = \partial\Phi/\partial z$ in $\mathcal{R}^i \backslash \mathcal{R}^c$ I computed the corresponding Stokes stream function Ψ by specifying

$$\left(\frac{\partial\Psi}{\partial r} + r u_z\right)\text{test}\left(\frac{\partial\Psi}{\partial r}\right) + \left(\frac{\partial\Psi}{\partial z} - r u_r\right)\text{test}\left(\frac{\partial\Psi}{\partial z}\right)$$

in the input field for a Weak Form PDE Physics interface. I set $\Psi = 0$ on $r = 0$ (all z) and accepted the default Null Flux boundary conditions on the core boundary and the portal.

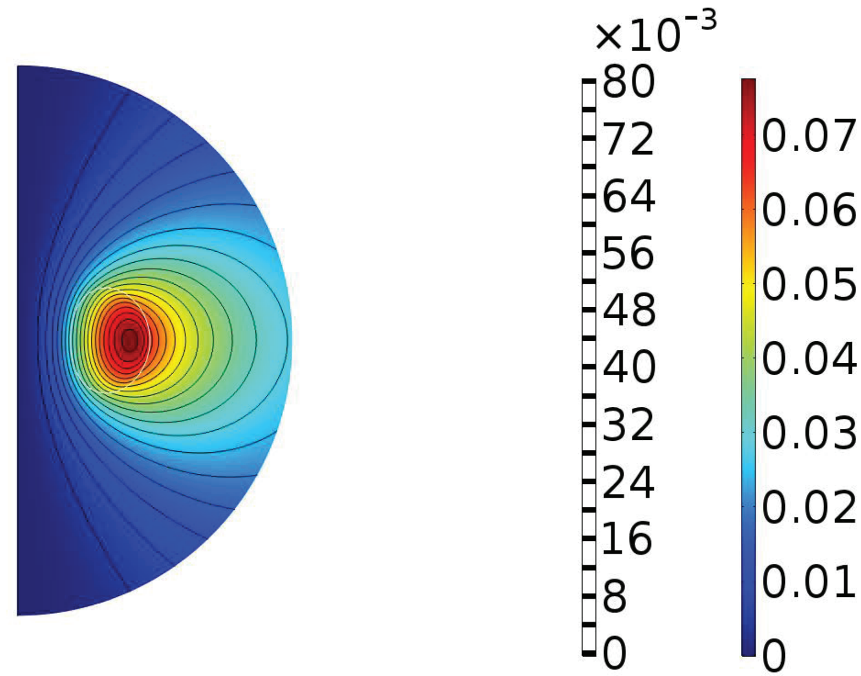


Figure 4 Ψ in $\mathcal{D}^i$ relative as seen by an observer at rest relative to the remote undisturbed fluid. The white contour is the core boundary. The increment of Ψ between contours is $0.04 \times 10^{-3}\, G/a^2$.

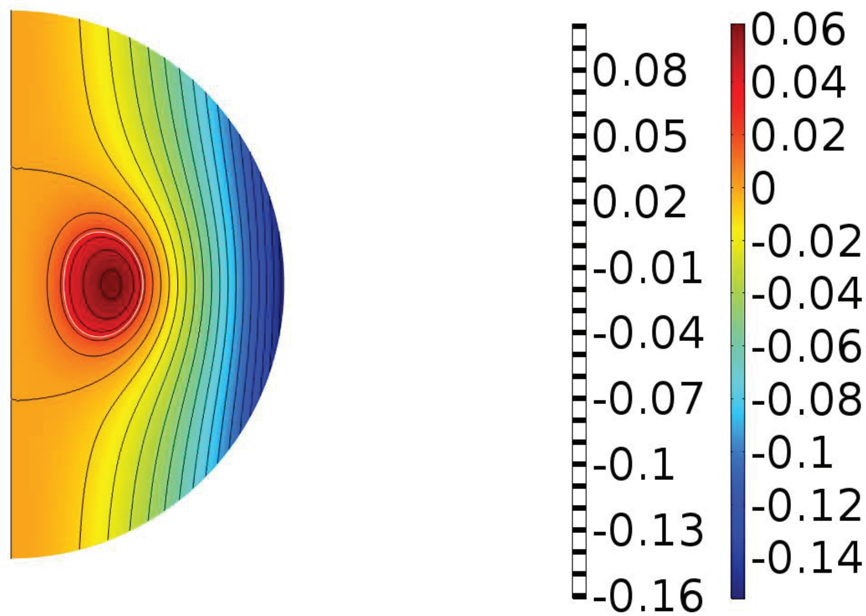


Figure 5 Legend similar to that of Fig. 4, except that now the results are as seen by an observer propagating with the ring.