Solving Distributed Optimal Control Problems of the Unsteady Burgers Equation in COMSOL Multiphysics

Fikriye Yılmaz
Department of Mathematics, Gazi University, Ankara-Turkey
Bülent Karasözen
Department of Mathematics & Institute of Applied Mathematics, Middle East Technical University, Ankara-Turkey

1 Problem Setting

- Burgers equation as a first approximation to complex diffusion convection phenomena and as simplified model for turbulence and in shock waves.
- Analysis and numerical approximation of optimal control problems for Burgers equation are important for the development of numerical methods for optimal control of more complicated models in fluid dynamics like Navier-Stokes equations.
- In contrast to linear parabolic control problems, the optimal control problem for the Burgers equation is a non-convex problem with multiple local minima due to nonlinearity of the differential equation. Numerical methods can only compute minima close to the starting points.

2 Optimal control of Burgers equations without inequality constraints

The distributed control problem without inequality constraints [7]:

$$\min \quad J(y,u) = \frac{1}{2}\|y - y_d\|_Q^2 + \frac{\alpha}{2}\|u\|_Q^2$$

$$\begin{aligned} \text{s.t.} \;\; y_t + y y_x - \nu y_{xx} &= f + bu \quad \text{in } Q, \\ y &= \;\; 0 \quad \text{on } \Sigma, \\ y(0) &= \;\; y_0 \quad \text{in } \Omega, \end{aligned}$$

y: state, u: control, y_d is the desired state.

$\Omega = (0,1)$, $T > 0$, $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \partial\Omega$.

First-order optimality conditions:

$$\begin{aligned} y_t - \nu y_{xx} + y y_x &= f + bu^* && \text{in } Q, \\ y(t,0) = y(t,1) &= 0 && \text{on } \Sigma, \\ y(0) &= y_0 && \text{in } \Omega, \end{aligned}$$

$$\begin{aligned} p_t + \nu p_{xx} + y p_x &= y_d - y^* && \text{in } Q, \\ p(t,0) = p(t,1) &= 0 && \text{on } \Sigma, \\ p(T) &= 0 && \text{in } \Omega, \end{aligned}$$

with the gradient condition $\alpha u^* + p = 0$.

u^*: the optimal control, $y*$: the associated optimal state, p: the adjoint state.

2.1 One-shot approach: treating the reverse time directions by simultaneous space-time discretization

- In the sequential approach optimality system is solved iteratively using the gradient method; the state equation was solved for y forwards and the adjoint equation backwards for p until convergence.
- In one-shot approach, the optimality system in the whole space-time cylinder is solved as an elliptic (biharmonic) equation by interpreting the time as an additional space variable [2].

$$\left.\begin{aligned} y_t + y y_x - \nu y_{xx} &= -\tfrac{1}{\alpha}p \\ p_t + \nu p_{xx} + y p_x &= y_d - y \end{aligned}\right\} \text{ in } Q,$$

$$\left.\begin{aligned} y &= 0 \\ p &= 0 \end{aligned}\right\} \text{ on } \Sigma,$$

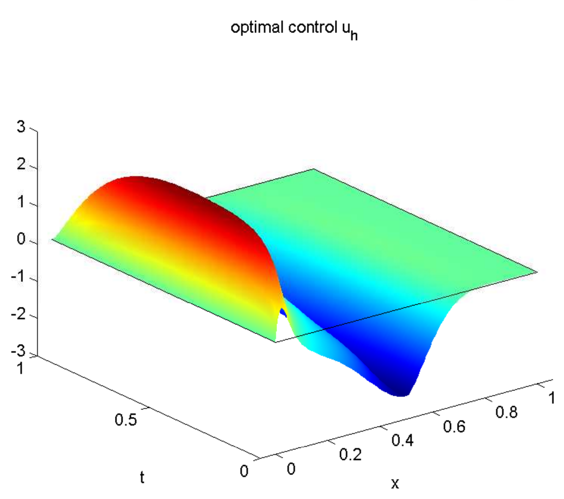
$$\begin{aligned} y &= y_0 && \text{in } \Omega \times \{0\}. \\ p &= 0 && \text{in } \Omega \times \{T\}. \end{aligned}$$

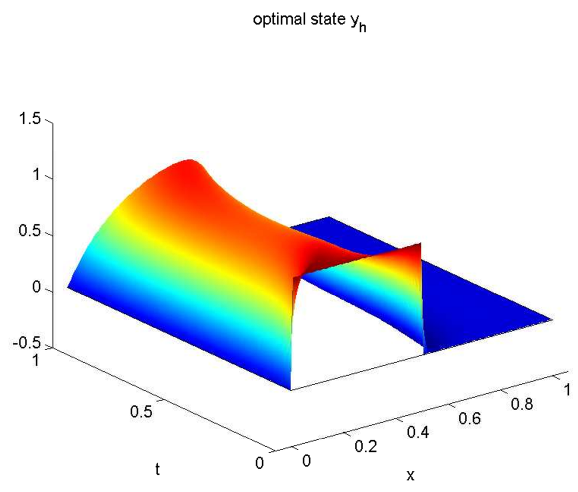
- Adaptive elliptic solver with ***adaption***.
- Nonadaptive elliptic solver ***femnlin***.
- Discretization of y and p by quadratic finite elements.

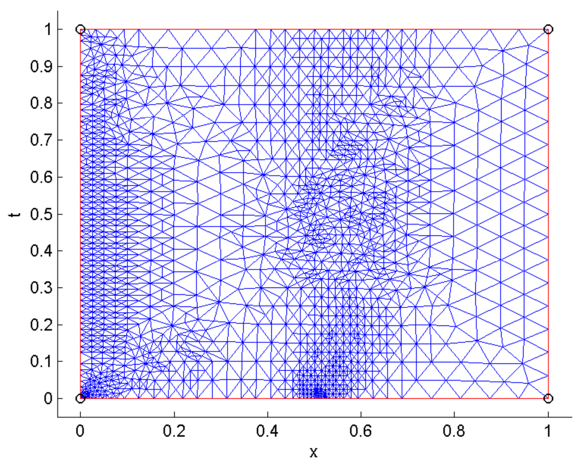
Numerical example [6]: $\alpha = 0.05$, $\nu = 0.01$, $f = 0$, the desired state $y_d(t,x) = y_0$ and the initial condition

$$y_0 = \begin{cases} 1 & \text{in } \left(0, \frac{1}{2}\right], \\ 0 & \text{otherwise.} \end{cases}$$

The control acts on the located support $(0,T) \times (\frac{1}{4}, \frac{3}{4})$.







Δx_{max}	$\|\|J(y,u)\|\|_Q$ with adaption	$\|\|J(y,u)\|\|_Q$ with femnlin
2^{-3}	0.0663	0.0651
2^{-4}	0.0667	0.0686
2^{-5}	0.0667	0.0671
2^{-6}	0.0667	0.0669

3 Optimal control of Burgers equation with inequality control constraints

Distributed optimal control problem with bilateral control constraints

$$\min \; J(y,u) = \frac{1}{2}\|y - z\|_Q^2 + \frac{\alpha}{2}\|u\|_Q^2$$

$$\begin{aligned} \text{s.t.} y_t + y y_x - \nu y_{xx} &= f + bu \text{ in } Q, \\ y &= 0 \text{ in } \Sigma, \\ y(,\cdot) &= y_0 \text{ in } \Omega, \end{aligned}$$

with pointwise control constraints [5]:

$$u_a(t,x) \le u(t,x) \le u_b(t,x) \text{ in } Q$$

The pointwise constraints leads to the variational inequality

$$\int_Q (\alpha u^* + b p^*)(u - u^*)\,dx\,dt \ge 0 \text{ for all } u \in U_{ad}.$$

which can be expressed in form of the projection:

$$u^*(t,x) = P_{[u_a(t,x),u_b(t,x)]}\left(\frac{-b(t,x)}{\alpha}p^*(t,x)\right)$$

Numerical example:
$(u \le u_b)$,$y_0 = \sin(13x)$, $\nu = 0.1$,$u_b = 0.3$, $\alpha = 0.01$, $y_d = y_0$ [4].

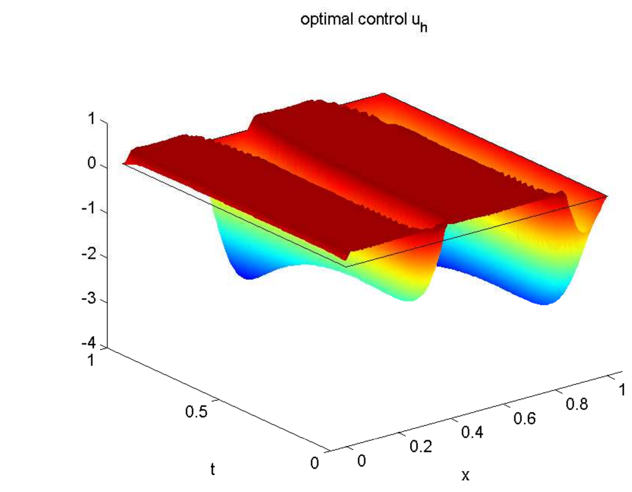
Implementation of the projection method in COMSOL Multiphysics (semi-smooth Newton method[1]):

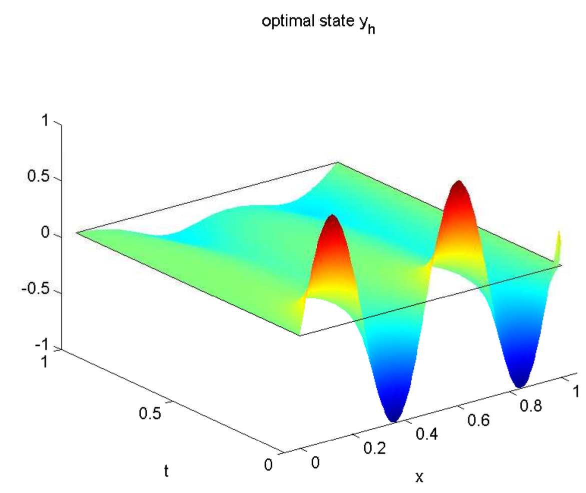
```
fem.equ.f= { {'-ytime-(p+mu)/alpha-yyx'
'ptime+y-zd(x,time)+y*px'...
'(1/alpha)*mu-max(0,-b-(1/alpha)*p)'}};
```

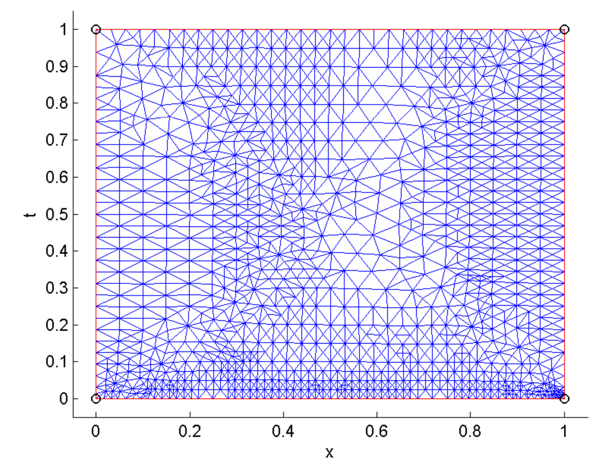
with the Lagrange multiplier $\mu = \alpha \max(0, -b - \frac{p}{\alpha})$ a.e. in Q.
The Lagrange multiplier μ is discretized by linear finite elements.

Solution of the the control constraint problem using the one-shot approach:

```
fem.form='general';
fem.globalexpr= {'u' '-(p+mu)/alpha' };
fem.equ.ga={{{'-nu*yx''0'}{'-nu*px ''0'}{'0''0'}}};
fem.equ.f= { {'-ytime-(p+mu)/alpha-y*yx'...
 'ptime+y-zd(x,time)+y*px' ...
 '(1/alpha)*mu-max(0,-0.3-(1/alpha)*p)' } };
fem.bnd.ind=[1 2 3 2];
fem.bnd.r= {{'y-y0(x)'0 0};{'y''p'0};{0'p'0}};
fem.bnd.g= { {0 0 0 }; {0 0 0 };{0 0 0 } };
postplot(fem,'tridata','y','triz','y')
```







Δx_{max}	$\|\|J(y,u)\|\|_Q$ with adaption	$\|\|J(y,u)\|\|_Q$ with femnlin
2^{-3}	0.2000	0.1985
2^{-4}	0.2002	0.2000
2^{-5}	0.2003	0.2002
2^{-6}	0.2003	0.2003

4 Conclusions and extensions

- We have shown that the finite element package of COMSOL Multiphysics can be used for solving time-dependent non-linear optimal control problems.
- Both classical gradient based approach solving the state equation forward in time and the adjoint equation backward in time and solving the the whole optimality system as an biharmonic equation produces satisfactory results for the Burgers equation.
- Considering Burgers equation with state constraints as it was done in [3] for parabolic control problems.
- Application of the various stabilization techniques available in COMSOL Multiphysics to the Burgers equation.

References

[1] M. Hintermüller, K. Ito, and K. Kunisch, *The Primal-Dual Active Set Strategy as a Semismooth Newton Method* , SIAM J. Optim. (2002), 865–888.

[2] I. Neitzel, U. Prüfert, and T. Slawig, *Solving time-dependent optimal control problems in COMSOL Multiphysics*, Proceedings of the COMSOL Conference Hannover, (2008).

[3] I. Neitzel, U. Prüfert, and T. Slawig, *Strategies for time-dependent pde control with inequality constraints using an integrated modeling and simulation environment*, Numer. Algorithms **50** (2009), 241–269.

[4] J. C. de los Reyes and K. Kunisch, *A comparison of algorithms for control constrained optimal control of the Burgers equation*, Calcolo **41** (2004), 203-225.

[5] F. Tröltzsch and S. Volkwein, *The SQP method for control constrained optimal control of the Burgers equation*, ESAIM Control Optim. Calc. Var **6** (2001), 649–674.

[6] S. Volkwein, *Distributed control problems for the Burgers equation*, Comput. Optim. Appl., **18** (2001), 115-140.

[7] S. Volkwein, *Mesh-independence of an augmented Lagrangian-SQP method in Hilbert spaces and control problems for the Burgers equation*, Diss. at the University of Technology Berlin, (1997).