FEM Convergence for PDEs with Point Sources in 2-D and 3-D

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Outline:

- Problem statement:
  Poisson equation in 2-D / 3-D with smooth / non-smooth forcing
- FEM Theory for Lagrange elements and regular mesh refinement
- Results: FEM convergence for smooth / non-smooth forcing

Poisson equation in 2-D / 3-D with smooth / non-smooth forcing

Poisson equation with Dirichlet boundary conditions
in $\Omega = (-1,1)^2$ for 2-D and in $\Omega = (-1,1)^3$ for 3-D

$$-\Delta u = f \quad \text{in } \Omega, \tag{1}$$

$$u = r \quad \text{on } \partial\Omega, \tag{2}$$

- Smooth forcing $f \in L^2(\Omega)$ and r in (1)–(2) such that PDE solution is

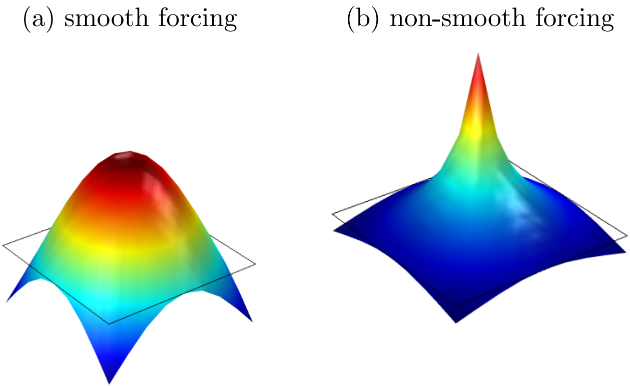
$$u(\mathbf{x}) = \cos\left(\frac{\pi}{2}\|\mathbf{x}\|_2\right) \quad \text{for } d = 2, 3 \tag{3}$$

- Non-smooth forcing $f = \delta \notin L^2(\Omega)$ and r in (1)–(2) such that PDE solution is

$$u(\mathbf{x}) = \begin{cases} \frac{-\ln\sqrt{x^2+y^2}}{2\pi} & \text{for } d = 2, \\ \frac{1}{4\pi\sqrt{x^2+y^2+z^2}} & \text{for } d = 3 \end{cases} \tag{4}$$

The Dirac delta distribution $\delta(\mathbf{x})$ models a point source at $\hat{\mathbf{x}} \in \Omega$ mathematically by requiring $\delta(\mathbf{x} - \hat{\mathbf{x}}) = 0$ for all $\mathbf{x} \neq \hat{\mathbf{x}}$, while simultaneously satisfying $\int_\Omega \varphi(\mathbf{x})\,\delta(\mathbf{x} - \hat{\mathbf{x}})\,d\mathbf{x} = \varphi(\hat{\mathbf{x}})$ for any continuous function $\varphi(\mathbf{x})$.

FEM Solution in 2-D Region $(-1,1) \times (-1,1)$



Three-dimensional view of the FEM solution in 2-D region $(-1,1) \times (-1,1)$ for the Poisson equation with (a) smooth forcing and (b) non-smooth forcing using linear Lagrange elements with 256 triangles.

FEM Theory for Lagrange Elements of Degree p

- Finite element solution u_h has error against PDE solution u of the form

$$\|u - u_h\|_{L^2(\Omega)} \leq C\, h^q, \quad \text{as } h \to 0, \tag{5}$$

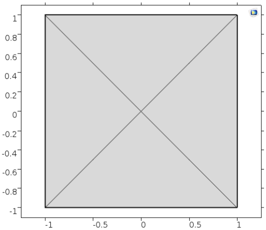
  where C is a problem-dependent constant independent of h and q is the convergence order of the FEM, as the mesh spacing h decreases.
- For Lagrange elements with piecewise polynomial degree p, the convergence order is $q = p + 1$.
- But the regularity order k of the Sobolev space of the PDE solution $u \in H^k(\Omega)$ limits the convergence order to $q = \min\{k, p + 1\}$ in (5).
- For the smooth problem, we have $k \geq 2$ in $u \in H^k(\Omega)$ and obtain the optimal convergence order of $q = p + 1$ for Lagrange elements with degree p. Specifically, $q = 2$ for linear Lagrange elements with $p = 1$.
- For the non-smooth problem, we have $k = 2 - d/2$ for $\Omega \subset \mathbb{R}^d$ and the convergence order q is limited to $q = 1.0$ for $d = 2$ and $q = 0.5$ for $d = 3$ for any Lagrange elements with degree $p \geq 1$.

2-D Mesh and Uniform Mesh Refinement

Finite element data for meshes in 2-D

- Uniform mesh refinement level m
- N_e = number of elements in mesh
- N = number of degrees of freedom (DOF)
- h = maximum side length of an element in the mesh
- The origin $\mathbf{x} = 0$ is enforced as a mesh point

Initial Mesh ($m = 0$)

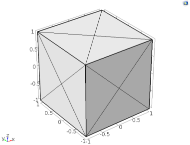


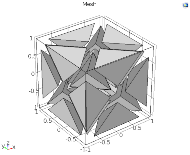
Mesh Refinement Data

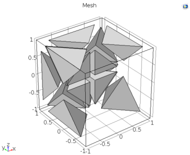
m	N_e	N = DOF	h
0	4	5	2.0000
1	16	13	1.0000
2	64	41	0.5000
3	256	145	0.2500
4	1,024	545	0.1250
5	4,096	2,113	0.0625

3-D Mesh and Uniform Mesh Refinement

Initial Mesh ($m = 0$)
with exploded view







Finite Element Data in 3-D

- Uniform mesh refinement level m
- N_e = number of elements in the mesh
- N = number of degrees of freedom
- h = maximum side length of an element
- The origin $\mathbf{x} = 0$ is enforced as a mesh point

m	N_e	N = DOF	h
0	28	15	2.0000
1	224	69	1.0000
2	1,792	409	0.5000
3	14,336	2,801	0.2500
4	114,688	20,705	0.1250
5	917,504	159,169	0.0625

Convergence Study with Smooth Forcing: 2-D and 3-D

Linear Lagrange elements ($p = 1$) for smooth test problem with m regular mesh refinements.
FEM error $E_m = \|u - u_h\|_{L^2}$ (convergence order $Q_m = \log_2(E_{m-1}/E_m)$):

2-D

m	E_m (Q_m)
0	1.105
1	3.049e–01 (1.86)
2	8.387e–02 (1.86)
3	2.177e–02 (1.95)
4	5.511e–03 (1.98)
5	1.383e–03 (1.99)

3-D

m	E_m (Q_m)
0	1.132
1	3.481e–01 (1.70)
2	9.007e–02 (1.95)
3	2.273e–02 (1.99)
4	5.690e–03 (2.00)
5	1.422e–03 (2.00)

- FEM error decreases with mesh refinement.
- Observed convergence order independent of dimension $q = 2$ in 2-D and 3-D in $\|u - u_h\|_{L^2} \leq C\, h^q$

Convergence Study with Non-Smooth Forcing in 2-D and 3-D

Linear Lagrange elements ($p = 1$) for non-smooth test problem with m regular mesh refinements.
FEM error $E_m = \|u - u_h\|_{L^2}$ (convergence order $Q_m = \log_2(E_{m-1}/E_m)$):

2-D

m	E_m (Q_m)
0	9.332e–02
1	4.589e–02 (1.02)
2	2.468e–02 (0.89)
3	1.256e–02 (0.97)
4	6.311e–03 (0.99)
5	3.160e–03 (1.00)

3-D

m	E_m (Q_m)
0	1.026e–01
1	6.990e–02 (0.55)
2	4.842e–02 (0.53)
3	3.410e–02 (0.51)
4	2.410e–02 (0.50)
5	1.704e–02 (0.50)

- FEM error decreases with mesh refinement
- Observed convergence order dimension-dependent
  $q = 1$ in 2-D and $q = 0.5$ in 3-D in $\|u - u_h\|_{L^2} \leq C\, h^q$

Conclusions and Recommendations

- COMSOL FEM solution converges according to FEM theory $\|u - u_h\|_{L^2} \leq C\, h^q$ as $h \to 0$ also in the presence of point source
- For smooth forcing $f \in L^2(\Omega)$, convergence order $q = 2$ for all dimensions $d = 2, 3$.
- For non-smooth forcing $f \notin L^2(\Omega)$, convergence order dimension-dependent: $q = 1$ in 2-D and $q = 0.5$ in 3-D.

Recommendations for COMSOL usage:

- Control initial mesh creation (very coarse) and its refinement (regular) explicitly; 3-D default refinement is: split longest side!
- Control the degree of the Lagrange elements explicitly
- Use linear Lagrange elements for problems involving point sources modeled by Dirac delta distributions
- Use higher-order elements (quadratic is default) for smooth problems