FEM Convergence for PDEs with Point Sources in 2-D and 3-D

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Introduction: We show how to demonstrate computationally the convergence order of the FEM in COMSOL. In the presence of a highly non-smooth point source modeled by the Dirac delta distribution convergence is present, but the order is limited. We control mesh creation and refinement explicitly. Linear Lagrange elements and regular mesh refinement are used.

Test Problems: Poisson equation with Dirichlet boundary conditions in $\Omega = (-1,1)^2$ for 2-D and in $\Omega = (-1,1)^3$ for 3-D.

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = r \quad \text{on } \partial\Omega$$

Smooth forcing: choose $f \in L^2(\Omega)$ and r such that the PDE solution is

$$u(\mathbf{x}) = \cos\left(\frac{\pi}{2}\|\mathbf{x}\|_2\right) \quad \text{in 2-D and 3-D}$$

Non-smooth forcing: choose $f = \delta \notin L^2(\Omega)$ and r such that the PDE solution is:

$$u(\mathbf{x}) = \begin{cases} \dfrac{-\ln\|\mathbf{x}\|_2}{2\pi} & \text{in 2-D} \\ \dfrac{1}{4\pi\|\mathbf{x}\|_2} & \text{in 3-D} \end{cases}$$

FEM error estimates have the form

$$\|u - u_h\|_{L^2} \le Ch^q \quad \text{as } h \to 0$$

The regularity order k of the PDE solution limits the convergence order q for linear Lagrange elements so that $q = \min\{k, 2\}$ in a dimension dependent way.

Conclusions:

- FEM solution converges according to FEM theory even in the presence of a point source using COMSOL.
- For smooth forcing, convergence is optimal and independent of dimension: $q = 2$ in 2-D and 3-D.
- For non-smooth forcing, convergence order is dimension dependent, in particular: $q = 1$ in 2-D and $q = 0.5$ in 3-D.

Reference: Proceedings of the COMSOL Conference 2015, Boston, MA.

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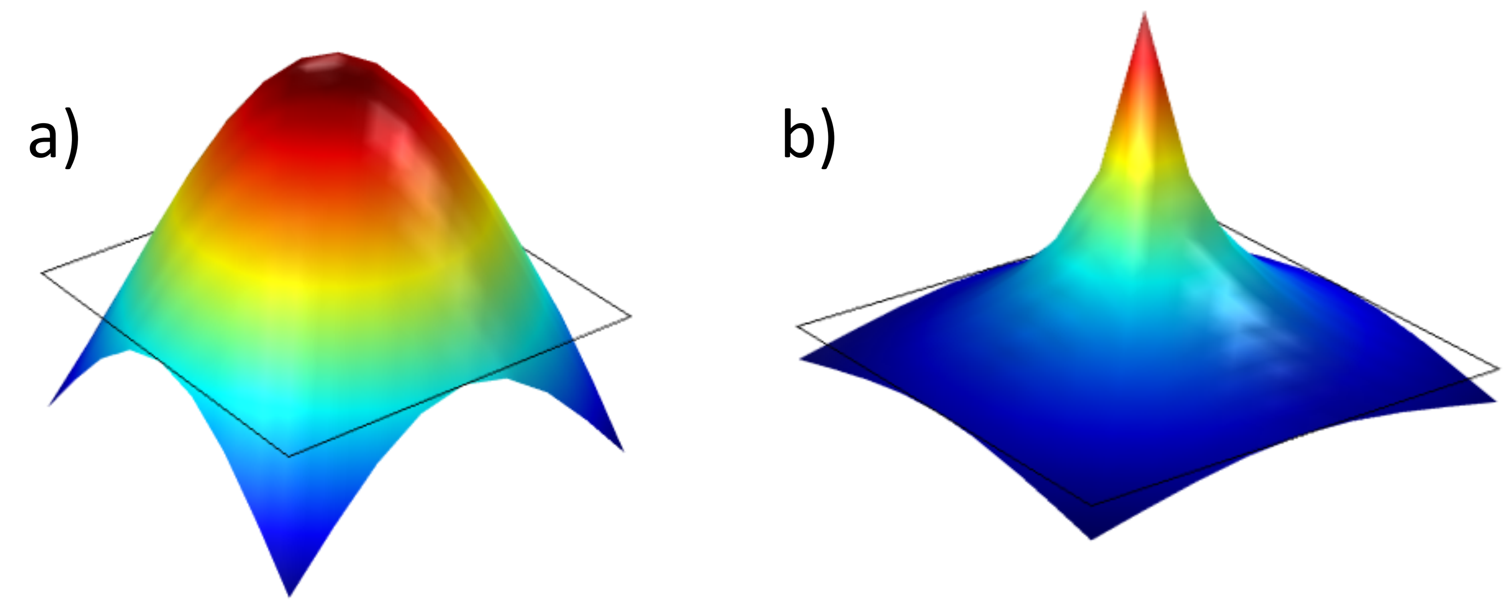

Figure 1. 3-D view of FEM solution on (-1,1) x (-1,1) with (a) smooth and (b) non-smooth forcing terms

Table 1.
Finite Element Mesh Data In 2-D (i) and 3-D (ii)

Regular mesh refinement level m
N_e = number of elements in mesh
N = number of degrees of freedom
h = maximum side length

(i) **2-D**

m	N_e	$N = DOF$	h
0	4	5	2.0000
1	5	13	1.0000
2	16	41	0.5000
3	256	145	0.2500
4	1,024	545	0.1250
5	4,096	2,113	0.0625

(ii) **3-D**

m	N_e	$N = DOF$	h
0	28	15	2.0000
1	224	69	1.0000
2	1,792	409	0.5000
3	14,336	2,801	0.2500
4	114,688	20,705	0.1250
5	917,504	159,169	0.0625

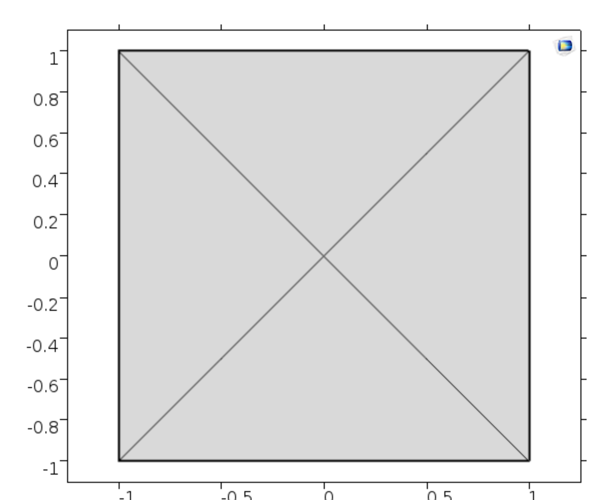

Figure 2. 2-D initial mesh

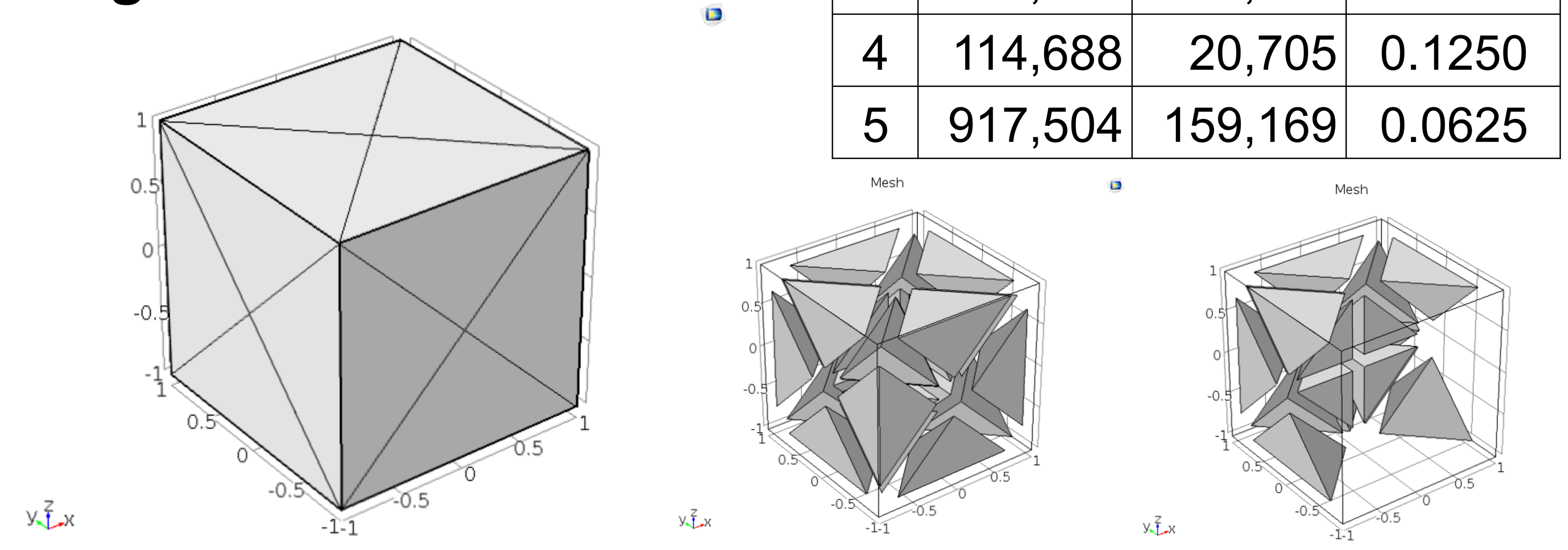

Figure 3. 3-D initial mesh with exploded views

Table 2. Convergence Studies

FEM error $E_m = \|u - u_h\|_{L^2}$ and
Estimated convergence order $Q_m = \log_2(E_{m-1}/E_m)$

Smooth **2-D**

m	$E_m(Q_m)$
0	1.105
1	3.049e-01 (1.86)
2	8.387e-02 (1.86)
3	2.177e-02 (1.95)
4	5.511e-03 (1.98)
5	1.383e-03 (1.99)

Smooth **3-D**

m	$E_m(Q_m)$
0	1.132
1	3.481e-01 (1.70)
2	9.007e-02 (1.95)
3	2.273e-02 (1.99)
4	5.690e-03 (2.00)
5	1.422e-03 (2.00)

Non-smooth **2-D**

m	$E_m(Q_m)$
0	9.332e-02
1	4.589e-02 (1.02)
2	2.468e-02 (0.89)
3	1.256e-02 (0.97)
4	6.311e-03 (0.99)
5	3.160e-03 (1.00)

Non-smooth **3-D**

m	$E_m(Q_m)$
0	1.026e-01
1	6.990e-01 (0.55)
2	4.842e-02 (0.53)
3	3.410e-02 (0.51)
4	2.410e-02 (0.50)
5	1.704e-02 (0.50)