Multigrid Implementation in COMSOL Multiphysics® - Comparison of Theory and Practice

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Abstract

Multigrid methods (MG) belong to the fastest solvers for partial differential equations. The key for this is an appropriate composition of the algorithmic components [1,2,4]. The multigrid solver implemented in COMSOL Multiphysics® is analyzed with respect to components and with respect to its numerical properties. Of special interest is the question whether solving selected model problems shows that behavior which is known from multigrid theory.

Experiments confirm that smoothing is a key component for MG. In COMSOL Multiphysics®, the standard methods are available. Variants with different types of scanning the equations are realized, too. Collective methods and block-relaxation methods can also be chosen. The grid hierarchy can be created by either coarsening from fine to coarse or by refining from coarse to fine, and other selections. The most relevant cycle types are implemented.

Exploiting all these possibilities allows the composition of MG algorithms, which possess the well-known MG behavior: convergence speed remains almost constant for every cycle, the convergence rate is essentially independent from the mesh size, the numerical effort per cycle is proportional to the problem size. The empirical convergence rates for model problems show, that COMSOL MG meets both theoretical prediction and empirical experience.

The V-cycle using Gauss-Seidel (SOR) creates a very fast (time to reach the tolerance) MG method for the Poisson model problem with an average convergence rate of 0.046 reaching the desired tolerance within 10 cycles.

It is demonstrated that for special situations where MG is known to badly converge or even to diverge (anisotropic Poisson equation and Poisson equation with Jacobi smoothing with relaxation factor equal to one, respectively) the COMSOL MG behaves like one should expect. In the first case the convergence rate is 0.915 (theory estimates 0.942, [3]) and the second case shows divergence. MG theory [2, 4] recommends a relaxation factor of 0.8 with Jacobi smoothing for the Poisson equation: with this modification of the default value COMSOL MG reaches a convergence factor of 0.193.

The advantage of choosing MG for solving the system of equations is also demonstrated when simulating the thermos holding hot coffee (from the model library) on a mesh with about 2.000.000 degrees of freedom: MUMPS and Pardiso solve within 146 and 80 seconds,

respectively. MG with V(2,1)-cycles, SOR-smoothing and totally 6 MG levels obtained by regular refinement solves the problem to the same accuracy within 50 seconds. Adding a further level of refinement (leading to more than 5.000.000 degrees of freedom) lead to 419, 226, and 119 seconds computing time for MUMPS, Pardiso, and MG, respectively. An additional level of regular refinement (more than 20.000.000 degrees of freedom) could be solved by MG within 648 seconds. The direct solver failed due to memory requirements (on a PC with 24 GByte memory).

Concluding: COMSOL Multiphysics® allows the composition of time-, convergence-, and memory-efficient MG algorithms both for model problems and concrete applications which work in the range predicted by theory and known by experience.

Reference

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3. R. Wienands, and W. Joppich: Numerical Insights into Local Fourier Analysis for Multigrid Methods, CRC Press, Boca Raton, 2004.
4. U. Trottenberg, et al.: Multigrid, Academic Press, San Diego, 2001.
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